Lect no-1	Subject Name: Mechanics of solids Prepared by: Kirti Ranjan Panda, Dept. Of Mechanical Engg,GCE,Keonjar

Load, Stress, Principle of St.Venant, Principle of Superposition, Strain, Hooke's law and related problem

Introduction:

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. This resistance by which material of the body opposes the deformation is known as strength of **material.** Within a certain limit (*i.e.*, in the elastic stage) the resistance offered by the material is proportional to the deformation brought out on the material by the external force. Also within this limit the resistance is equal to the external force (or applied load). But beyond the elastic stage, the resistance offered by the material is less than the applied load. In such a case, the deformation continues, until failure takes place.

Within elastic stage, the resisting force equals applied load. This resisting force per unit area is called stress or intensity of stress.

Load: It is defined as any external force acting upon a machine part.ther are four types of load

1.**Dead and steady load**: load is said to be dead and steady load ,when it does not change magnitude or direction

2.Live or variable load : load is said to be live or variable load ,when it changes continually

3.Suddenlly applied or shock load : A load is said to be suddenly applied load ,when it is suddenly applied or removed.

4. Impact load : A load is said to be impact load ,when it is applied with some initial velocity

STRESS

The force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called the *load or force*. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

Mathematically stress is written as, $\sigma = \frac{P}{A}$

where σ = Stress (also called intensity of stress),

P = External force or load, and

A = Cross-sectional area.

Types of stress

The stress may be normal stress or a shear stress.

Normal stress is the stress which acts in a direction perpendicular to the area. It is represented by σ (sigma). The normal stress is further divided into tensile stress and compressive stress.

Tensile Stress. The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig. 1.1 (*a*) as a result of which there is an increase in length, is known as tensile stress. The ratio of increase in length to the original length is known as *tensile strain*. The tensile stress acts normal to the area and it pulls on the area.

Let P = Pull (or force) acting on the body,
A = Cross-sectional area of the body,
L = Original length of the body,
dL = Increase in length due to pull P acting on the body,
σ = Stress induced in the body, and
e = Strain (*i.e.*, tensile strain).

Fig. 1.1 (*a*) shows a bar subjected to a tensile force P at its ends. Consider a section x-x, which divides the bar into two parts. The part left to the section x-x, will be in equilibrium if P = Resisting force (R). This is shown in Fig. 1.1 (*b*). Similarly the part right to the section x-x, will be in equilibrium if P = Resisting force as shown in Fig. 1.1 (*c*). This resisting force per unit area is known as stress or intensity of stress.

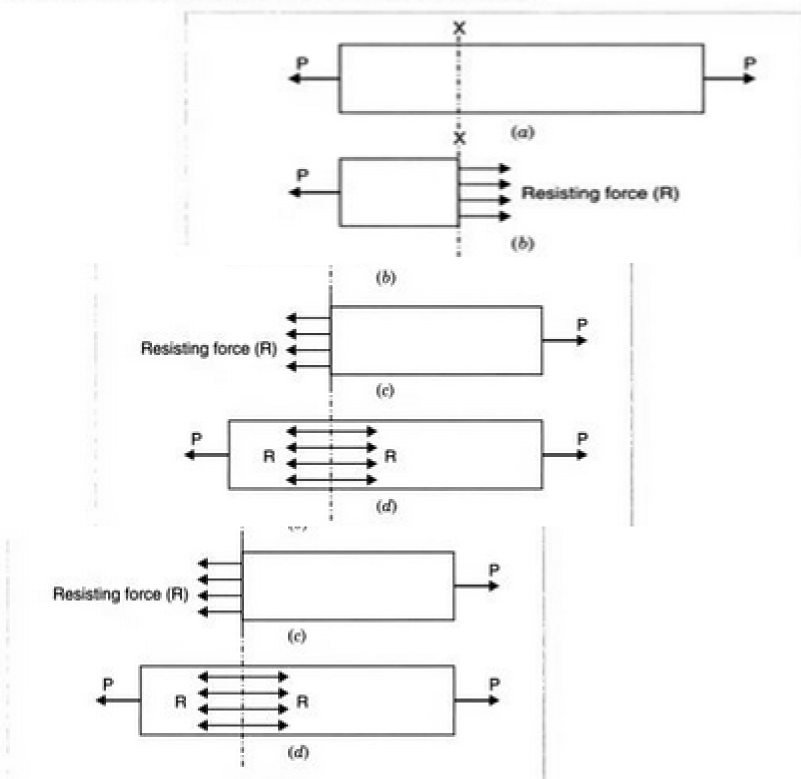


Fig. 1.1

∴ Tensile stress $= \sigma = \dfrac{\text{Resisting force } (R)}{\text{Cross-sectional area}} = \dfrac{\text{Tensile load } (P)}{\text{A}}$

$$\sigma = \frac{P}{A}$$

And tensile strain is given by,

$$e = \frac{\text{Increase in length}}{\text{Original length}} = \frac{dL}{L}.$$

Compressive Stress. The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig. 1.2 (a) as a result of which there is a decrease in length of the body, is known as compressive stress. And the ratio of decrease in length to the original length is known as *compressive strain*. The compressive stress acts normal to the area and it pushes on the area.

Let an axial push P is acting on a body in cross-sectional area A. Due to external push P, let the original length L of the body decreases by dL.

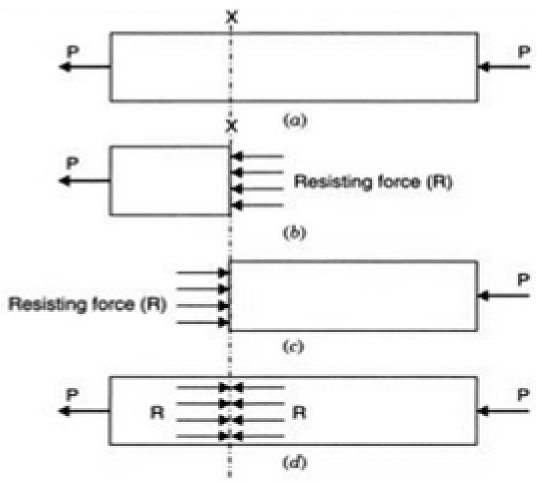


Fig. 1.2

Then compressive stress is given by,

$$\sigma = \frac{\text{Resisting Force }(R)}{\text{Area }(A)} = \frac{\text{Push }(P)}{\text{Area }(A)} = \frac{P}{A}.$$

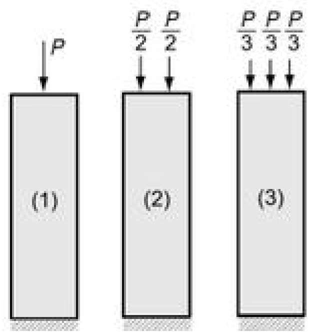
Shear Stress. The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as shown in Fig. 1.3 as a result of which the body tends to shear off across the section, is known as shear stress. The corresponding strain is known as *shear strain*. The shear stress is the stress which acts tangential to the area. It is represented by τ.

Principle of St.Venant

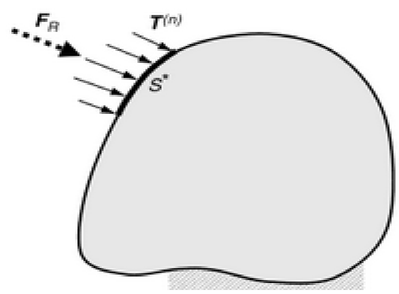
Statically Equivalent Loadings.

Consider the set of three identical rectangular strips under compressive loadings as shown in Figure 5.9. As indicated, the only difference between each problem is the loading. Because the total resultant load applied to each problem is identical (statically equivalent loadings), it is expected that the resulting stress, strain, and displacement fields near the bottom of each strip would be approximately the same.

This behavior can be generalized by considering an elastic solid with an arbitrary loading $\boldsymbol{T}^{(n)}$ over a boundary portion S^*, as shown in Figure 5.10. Based on experience from other field problems in engineering science, it seems logical that the particular boundary loading would produce detailed and



Statically Equivalent Loadings.



Saint-Venant's Principle.

characteristic effects only in the vicinity of S^*. In other words, we expect that at points far away from S^* the stresses generally depend more on the *resultant* $\boldsymbol{F}_R$ of the tractions rather than on the exact distribution. Thus, the *characteristic signature* of the generated stress, strain, and displacement fields from a given boundary loading tend to disappear as we move away from the boundary loading points. These concepts form the *principle of Saint-Venant*, which can be stated as follows:

***Saint-Venant's Principle:** The stress, strain, and displacement fields caused by two different statically equivalent force distributions on parts of the body far away from the loading points are approximately the same.*

Principle of Superposition. When a number of loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained. The total deformation of the body will be then equal to the algebraic sum of deformations of the individual sections.

Analysis bars of varying section

A bar of different lengths and of different diameters (and hence of different cross-sectional areas) is shown in Fig. 1.6 (*a*). Let this bar is subjected to an axial load *P*.

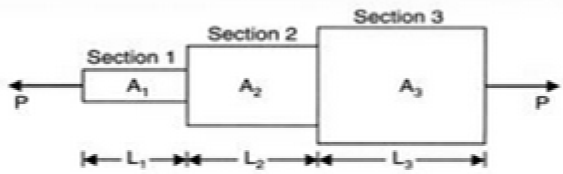


Fig. 1.6 (*a*)

Though each section is subjected to the same axial load *P*, yet the stresses, strains and change in lengths will be different. The total change in length will be obtained by adding the changes in length of individual section.

Let P = Axial load acting on the bar,

L_1 = Length of section 1,

A_1 = Cross-sectional area of section 1,

L_2, A_2 = Length and cross-sectional area of section 2,

L_3, A_3 = Length and cross-sectional area of section 3, and

E = Young's modulus for the bar.

Then stress for the section 1,

$$\sigma_1 = \frac{\text{Load}}{\text{Area of section 1}} = \frac{P}{A_1} .$$

Similarly stresses for the section 2 and section 3 are given as,

$$\sigma_2 = \frac{P}{A_2} \quad \text{and} \quad \sigma_3 = \frac{P}{A_3}$$

Using equation (1.5), the strains in different sections are obtained.

$$\therefore \quad \text{Strain of section 1, } e_1 = \frac{\sigma_1}{\Sigma} = \frac{P}{A_1 E} \qquad \left(\because \ \sigma_1 = \frac{P}{A_1}\right)$$

Similarly the strains of section 2 and of section 3 are,

$$e_2 = \frac{\sigma_2}{E} = \frac{P}{A_2 E} \quad \text{and} \quad e_3 = \frac{\sigma_3}{E} = \frac{P}{A_3 E} .$$

$$\text{But strain in section 1} = \frac{\text{Change in length of section 1}}{\text{Length of section 1}}$$

or

$$e_1 = \frac{dL_1}{L_1}$$

where dL_1 = change in length of section 1.

$$\therefore \quad \text{Change in length of section 1, } dL_1 = e_1 L_1$$

$$= \frac{PL_1}{A_1 E} \qquad \left(\because \ e_1 = \frac{P}{A_1 E}\right)$$

Similarly changes in length of section 2 and of section 3 are obtained as :

Change in length of section 2, $dL_2 = e_2 L_2$

$$= \frac{PL_2}{A_2 E} \qquad \left(\because \ e_2 = \frac{P}{A_2 E}\right)$$

and change in length of section 3, $dL_3 = e_3 L_3$

$$= \frac{PL_3}{A_3 E} \qquad \left(\because \ e_3 = \frac{P}{A_3 E}\right)$$

$\therefore$ Total change in the length of the bar,

$$dL = dL_1 + dL_2 + dL_3 = \frac{PL_1}{A_1 E} + \frac{PL_2}{A_2 E} + \frac{PL_3}{A_3 E}$$

$$= \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right]$$

Equation (1.8) is used when the Young's modulus of different sections is same. If the Young's modulus of different sections is different, then total change in length of the bar is given by,

$$dL = P\left[\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} + \frac{L_3}{E_3 A_3}\right] \qquad ...(1.9)$$

Strain

When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless.

Strain may be :

1. Tensile strain,
2. Compressive strain,
3. Volumetric strain, and
4. Shear strain.

If there is some increase in length of a body due to external force, then the ratio of increase of length to the original length of the body is known as *tensile strain*. But if there is some decrease in length of the body, then the ratio of decrease of the length of the body to the original length is known as *compressive strain*. The ratio of change of volume of the body to the original volume is known as *volumetric strain*. The strain produced by shear stress is known as shear strain.

Hook's law

Hooke's Law states that when a material is loaded within elastic limit, the stress is proportional to the strain produced by the stress. This means the ratio of the stress to the corresponding strain is a constant within the elastic limit. This constant is known as Modulus of Elasticity or Modulus of Rigidity or Elastic Modulii.

Lect -2
Modulus of Elasticity, Stress-Strain Diagrams, Working Stress

Modulus of Elasticity

The ratio of tensile stress or compressive stress to the corresponding strain is a constant. This ratio is known as Young's Modulus or Modulus of Elasticity and is denoted by E.

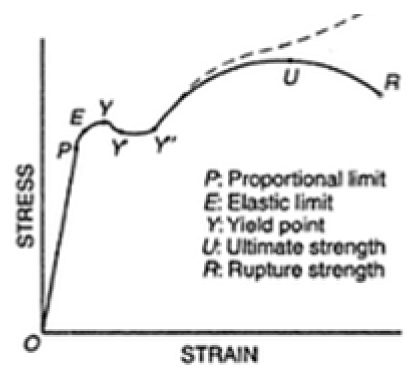
$$\therefore \qquad E = \frac{\text{Tensile stress}}{\text{Tensile strain}} \quad \text{or} \quad \frac{\text{Compressive stress}}{\text{Compressive strain}}$$

Stress-Strain Diagrams

The behaviour of a material subjected to an increased tensile load is studied by testing a specimen in a tensile testing machine and plotting the stress–strain diagram. Stress–strain diagrams of different materials vary widely. However, it is possible to distinguish some common characteristics among various stress–strain diagrams of various groups of materials. It is observed that broadly the materials can be divided into two categories on the basis of these characteristics: ductile materials and brittle materials.

Ductile materials such as steel and many alloys of other metals have the ability to yield at normal temperatures. The plot between strain and the corresponding stress of a ductile material is represented graphically by a tensile test diagram. Figure 1.58 shows a stress vs. strain diagram for steel in which the stress is calculated on the basis of original area of a steel bar. Most of other engineering materials show a similar pattern to a varying degree. The following are the salient features of the diagram:

- When the load is increased gradually, the strain is proportional to load or stress upto a certain value. Line OP indicates this range and is known as the *line of proportionality*. Hooke's law is applicable in this range. The stress at the end point P is known as the *proportional limit*.
- If the load is increased beyond the limit of proportionality, the elongation is found to be more rapid, though the material may still be in the elastic state, i.e., on removing the load, the strain vanishes. This elongation with a relatively small increase in load is caused by slippage of the material along oblique surfaces



and is mainly due to shear stresses. The point E depicts the elastic limit. Hooke's law cannot be applied in this range as the strain is not proportional to stress. Usually, this point is very near to P and many times the difference between P and E is

- When the load is further increased, plastic deformation occurs, i.e., on removing the load, the strain is not fully recoverable. At point Y, metal shows an appreciable strain even without further increasing the load. Actually, the curve drops slightly at this point to Y' and the yielding goes up to the point Y''. The points Y' and Y'' are known as the *upper* and *lower yield points* respectively. The stress–strain curve between Y and Y'' is not steady.
- After the yield point, further straining is possible only by increasing the load. The stress–strain curve rises up to the point U, the strain in the region Y to U is about 100 times that from O to Y'. The stress

- If the bar is stressed further, it begins to form a *neck*, or a local reduction in cross-section occurs. After this, somewhat lower loads are sufficient to keep the specimen elongating further. Ultimately, the specimen fractures at point R. It is noted that fracture occurs along a cone-shaped surface at about 45° with the original surface of the specimen indicating that shear is primarily responsible for the failure of ductile materials.
- If the load is divided by the original area of the cross-section, the stress is known as the *nominal stress*. This is lesser at the rupture load than at the maximum load. However, the stress obtained by dividing with the reduced area of cross-section is known as the *actual* or *true* stress and is greater at the maximum load. It is shown in the figure by the dotted line.

In brittle materials such as cast iron, glass and stone, etc., rupture occurs without any appreciable change in the rate of elongation and there is no difference between the ultimate strength and the rupture strength. The strain at the time of rupture is much smaller for brittle materials as compared to ductile materials. There is no neck formation of the specimen of a brittle material and the rupture occurs along a perpendicular surface to the load indicating that normal stresses are primarily responsible for the failure of brittle materials

Working Stress

When the stress of the material lower than the maximum or ultimate stress at which failure of material take place ,the stress is known as the working stress or design stress .it is also known as safe or allowable stress

Lect-3

Factor of safety, Strain energy in tension and compression, Resilience

Factor of safety

It is defined as the ratio of ultimate tensile stress to the working (or permissible) stress. Mathematically it is written as

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Permissible stress}} \qquad ...(1.7)$$

Strain energy in tension and compression, Resilience

During loading a specimen, stress and strain are developed in the specimen depending upon the type of the load applied.

$$\text{Stress} = \sigma = \frac{P}{\text{area}} \qquad (3\text{-}24)$$

$$\text{Strain} = \epsilon = \frac{dl}{l} \qquad (3\text{-}25)$$

During loading, work is done on the specimen and this work is converted into strain energy. This strain energy, within the elastic limit, is known as resilience (Fig. 3-12). Machine members, like helical, spiral and leaf springs, are used in machines because of their property of resilience. During unloading, this energy is fully recoverable. During loading, the spring absorbs strain energy and when the load is removed from the spring, the stored energy is fully released. In the case of internal combustion engines, when the valve is opened, the spring on the stem of the valve is compressed, and when the spring releases the energy, the valve is closed.

$$\text{Strain energy per unit volume} = u = \frac{1}{2}\sigma\epsilon = \frac{\sigma^2}{2E}\ \text{(resilience)} \qquad (3\text{-}26)$$

$$\text{Total strain energy} \quad U = \frac{\sigma^2}{2E} \times \text{volume} \qquad (3\text{-}27)$$

The maximum strain energy absorbed by a body until it reaches its elastic limit is known as *proof resilience*, and the proof resilience per unit volume, is known as ***modulus** of resilience*.

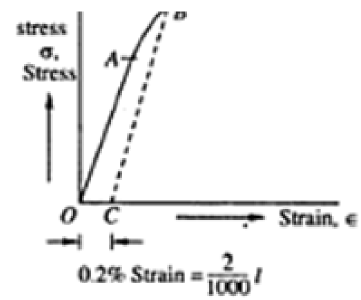


Figure 3-11 Stress vs. strain showing 0.2% proof stress

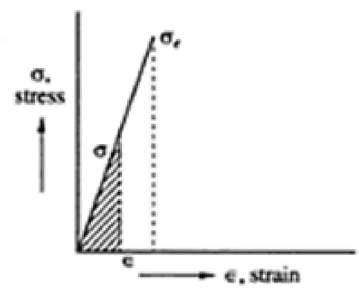


Figure 3-12 Stress vs. strain showing strain energy within elastic limit

$$\text{Proof resilience} = U_p = \frac{\sigma_e^2}{2E} \times \text{volume} \qquad (3\text{-}28)$$

Proof Resilience

The strain energy per unit volume of the material is known as resilience.

$\therefore$ Resilience, $$u = \frac{1}{V} \times U = \frac{p^2}{2E} = \frac{1}{2} p \cdot e = \frac{1}{2} E e^2 \qquad ...(6.7)$$

Resilience is also known as *strain energy density*. It represents the ability of the material to absorb energy within elastic limit.

When the stress p is equal to *proof stress* f at the elastic limit, the corresponding resilience is known as *proof resilience*

$\therefore$ $$u_p = \frac{f^2}{2E} \qquad ...(6.7\ a)$$

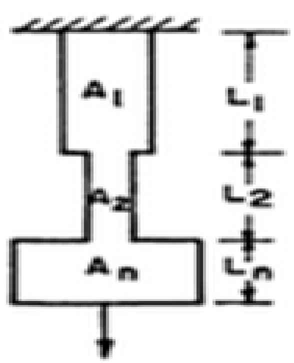
The proof resilience, also known as *modulus of resilience*, may be looked upon as the property of the material. The units of resilience are in Joules/m³ (N-m/m³ = N/m²).

The term resilience is sometimes employed to denote total quantity of strain energy stored but in specific problems the implication is obvious.

STRAIN ENERGY OF PRISMATIC BARS WITH VARYING SECTIONS

Fig. 6.4 shows a prismatic bar with varying section along its length.
In general, from Eq. 6.5

$$U = \frac{P^2 L}{2AE}$$

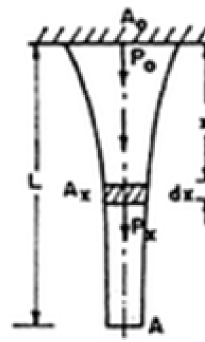


$\therefore$ Total $$U = \Sigma \frac{P^2 L}{2AE} = \frac{P^2}{2E} \left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_n}{A_n} \right]$$

STRAIN ENERGY OF NON-PRISMATIC BAR WITH VARYING AXIAL FORCE

In general strain energy of a bar is given by Eq. 6.5

$$U = \frac{P^2 L}{2AE}$$



Applying this to a differential section of length dx, where the axial force is P_x, we have

$$U = \int_0^L \frac{P_x^2\, dx}{2EA_x} \qquad ...(6.9)$$

where A_x is the area of cross-section of the differential section.

STRAIN ENERGY OF PRISMATIC BAR HANGING UNDER ITS OWN WEIGHT

Consider an element of length dx, at distance x from the support A, for a bar hang freely under its own weight. Assume that elastic conditions prevail.

Consider an element of length dx, at distance x from the support A, for a bar hang freely under its own weight. Assume that elastic conditions prevail.

Axial force P_x below the section is

$$P_x = \gamma A (L - x)$$

where γ is the specific weight of the material.

Now for the element, $U_x = \dfrac{P_x^2\, dx}{2AE}$

$\therefore$ Total strain energy,

$$U = \int_0^L \frac{P_x^2\, dx}{2AE} = \int_0^L \frac{[\gamma A (L - x)]^2\, dx}{2AE}$$

$$= \frac{\gamma^2 A L^3}{6E} \qquad ...(6.10)$$

Alternatively, stress $\quad p_x = \dfrac{P_x}{A} = \gamma (L - x)$

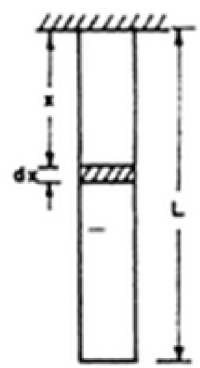


FIG. 6.6

Strain energy density of the element, $u_x = \dfrac{p_x^2}{2E} = \dfrac{\gamma^2 (L-x)^2}{2E}$

$$\therefore \qquad U = \int u_x\, dV = \int_0^L \frac{\gamma^2 (L-x)^2}{2E} (A \,.\, dx) = \frac{\gamma^2 A L^3}{6E}$$

STRAIN ENERGY OF FREELY HANGING PRISMATIC BAR WITH AN AXIAL LOAD

As found earlier, axial force below the sections, due to self weight $= \gamma A (L - x)$

$\therefore$ Total $\qquad P_x = \gamma A (L - x) + P$

$$\therefore \quad U = \int_0^L \frac{[\gamma A (L - x) + P]^2 dx}{2AE} = \frac{\gamma^2 A L^3}{6E} + \frac{P^2 L}{2AE} + \frac{\gamma P L^2}{2E} \quad ...(6.11)$$

$$= U_1 + U_2 + U_3$$

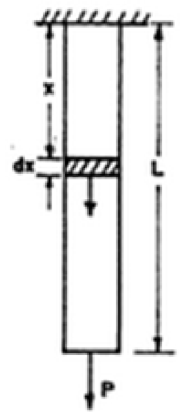


FIG. 6.7

The above result shows that the total strain energy stored in the bar consists of *three* components :

(*i*) U_1 = strain energy due to a freely hanging bar (Eq. 6.10)

(*ii*) U_2 = Strain energy due to an axial load (Eq. 6.5), and

(*iii*) U_3 = Strain energy component which is a function of both P as well as γ.

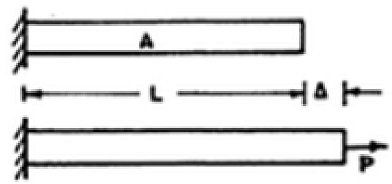
This shows that the strain energy of an elastic body due to more than one load cannot be found merely by adding the strain energies obtained from individual loads. This is because of the fact that strain is *not* a linear function but is a quadratic function of the loads, as is evident from Eq. 6.9.

Lect-4
Impact loads and related problem Composite bars in tension and compression

STRESSES DUE TO GRADUAL, SUDDEN AND IMPACT LOADINGS

(a) Gradual Loading

Let a bar of cross-sectional area A be subjected to a gradually applied axial load P due to which the deformation is Δ.



Then work done by external load

$$= \frac{1}{2} P . \Delta$$

Work stored in the body $= \frac{1}{2} R \Delta = \frac{1}{2} p A \Delta$

(where R is the resistance set up by the body and p is the stress induced)

Equating the work stored to the work done, we get

$$\frac{1}{2} p A \Delta = \frac{1}{2} P \Delta$$

From which $\quad p = \frac{P}{A}$ ...(6.12)

Thus, the *maximum stress set up in the bar is equal to the load divided by the area of cross-section.*

(b) Sudden Loading

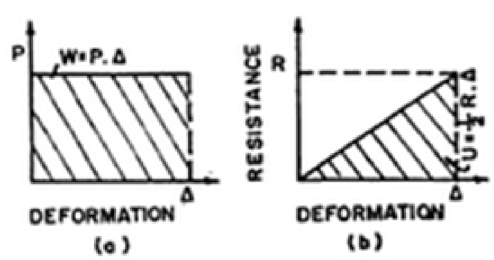
If the load P is applied suddenly (instead of being applied gradually), the value of load is P throughout the deformation. (Fig. 6.12 *a*). The deformation, however increases from zero to its final value Δ.

$\therefore$ Work done on the bar $= P . \Delta$ ...(*i*)

However, the resistance R set up in the body is zero when the deformation is zero, and is equal to R when the deformation is Δ. (Fig. 6.12 *b*). Hence work stored

$$= U = \frac{1}{2} R \Delta$$

$$= \frac{1}{2} p . A . \Delta \qquad ...(ii)$$



Equating the two ,

$$\frac{1}{2} p A \Delta = P \Delta$$

or $$p = \frac{2P}{A} \qquad ...(6.13)$$

Thus the *maximum stress set up in the body is equal to twice that in the case of gradual loading*.

(c) Impact Loading

Let us now consider the impact loading. Fig. 6.13 shows load P dropping on the collar of an elastic body through the height h, before it commences to stretch.

As the weight P, after falling through the height h, strikes the collar fixed at the lower end of the bar, small oscillation is set up initially, provided elastic limit is not exceeded. After that, the collar will take up some final position, as in the case of gradually applied load. Let the final deformation be Δ.

Let us assume that weight and the supports of the bar are infinitely rigid, so that the whole energy in the falling weight is expanded in stretching the bar with an amount Δ.

Now, work done by falling weight P is

Equating the two, we get $P\left(h + \frac{pL}{E}\right) = \frac{p^2}{2E} A L$

Rearranging, $p^2\left(\frac{AL}{2E}\right) - p\left(\frac{PL}{E}\right) - Ph = 0$

Dividing all the terms by $\frac{AL}{E}$, we get

$$\frac{p^2}{2} - \left(\frac{P}{A}\right)p - \frac{PEh}{AL} = 0$$

From which $$p = \frac{P}{A} \pm \sqrt{\left(\frac{P}{A}\right)^2 + \left(4 \times \frac{1}{2}\right)\left(\frac{PEh}{AL}\right)}$$

or $$P = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{PL}}\right) \qquad ...(6.14)$$

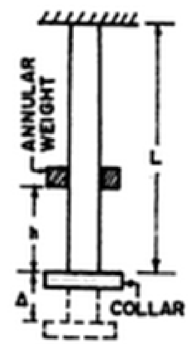


FIG. 6.13

(The positive root giving the maximum stress)

If, however, Δ is considered negligible in companion to h, we have

$$P.h = \frac{p^2}{2E}.AL$$

or $$p = \sqrt{\frac{2EPh}{AL}} \qquad ...(6.15)$$

Let us express the above equations in terms of deformation Δ, noting that $W = P\left(h + \Delta\right)$ and $U = \frac{1}{2}P\Delta$

$$\therefore \quad P(h + \Delta) = \frac{1}{2}P\Delta$$

But $$\Delta = \frac{PL}{AE} \text{ (Hooke's Law) or } P = \frac{\Delta E A}{L}$$

$$\therefore \quad P(h + \Delta) = \frac{EA\Delta^2}{2L}$$

Multiplying all the sides by $\frac{2L}{EA}$

We get $$\Delta^2 - \frac{2PL}{EA}\Delta - \frac{2PLh}{EA} = 0,$$ which is quadratic equation for Δ.

Solving, we get $$\Delta = \frac{PL}{AE} + \sqrt{\left(\frac{PL}{AE}\right)^2 + \frac{2PLh}{AE}} \quad ...(6.16)$$

Now introducing $\Delta_{st} = \frac{PL}{AE} =$ static deflection of bar due to P,

Eq. 6.16 becomes $$\Delta = \Delta_{st} + \sqrt{(\Delta_{st})^2 + (2h\,\Delta_{st})} \quad ...(6.17)$$

If Δ_{st} is considered very small in comparison to h, we get

$$\Delta \approx \sqrt{2h\,\Delta_{st}} \quad ...(6.18)$$

This could also be obtained from Eq. 6.15, by putting $p = \frac{P}{A} = \frac{\Delta E}{L}$

Thus, $$\frac{\Delta E}{L} = \sqrt{\frac{2E^2h}{L^2}\left(\frac{PL}{AE}\right)} = \frac{E}{L}\sqrt{2h\,\Delta_{st}}$$

or $$\Delta = \sqrt{2h\,\Delta_{st}}$$

The ratio $\frac{\Delta}{\Delta_{st}}$ is called the *impact factor*.

Composite bars in tension and compression

A bar, made up of two or more bars of equal lengths but of different materials rigidly fixed with each other and behaving as one unit for extension or compression when subjected to an axial tensile or compressive loads, is called a composite bar. For the composite bar the following two points are important :

1. The extension or compression in each bar is equal. Hence deformation per unit length *i.e.*, strain in each bar is equal.

2. The total external load on the composite bar is equal to the sum of the loads carried by each different material.

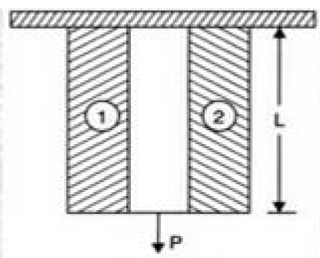


Fig. 1.15

Fig. 1.15 shows a composite bar made up of two different materials.

Let P = Total load on the composite bar,

L = Length of composite bar and also length of bars of different materials,

A_1 = Area of cross-section of bar 1,

A_2 = Area of cross-section of bar 2,

E_1 = Young's Modulus of bar 1,

E_2 = Young's Modulus of bar 2,

P_1 = Load shared by bar 1,

P_2 = Load shared by bar 2,

σ_1 = Stress induced in bar 1, and

σ_2 = Stress induced in bar 2.

Now the total load on the composite bar is equal to the sum of the load carried by the two bars.

$$P = P_1 + P_2$$

The stress in bar 1, $= \frac{\text{Load carried by bar 1}}{\text{Area of cross-section of bar 1}}$.

$\therefore$ $$\sigma_1 = \frac{P_1}{A_1} \quad \text{or} \quad P_1 = \sigma_1 A_1 \quad ...(ii)$$

Similarly stress in bar 2, $$\sigma_2 = \frac{P_2}{A_2} \quad \text{or} \quad P_2 = \sigma_2 A_2 \quad ...(iii)$$

Substituting the values of P_1 and P_2 in equation (*i*), we get

$$P = \sigma_1 A_1 + \sigma_2 A_2 \quad \text{...}(iv)$$

Since the ends of the two bars are rigidly connected, each bar will change in length by ie same amount. Also the length of each bar is same and hence the ratio of change in length the original length (*i.e.*, strain) will be same for each bar.

But strain in bar 1, $= \dfrac{\text{Stress in bar 1}}{\text{Young's modulus of bar 1}} = \dfrac{\sigma_1}{E_1}$.

Similarly strain in bar 2, $= \dfrac{\sigma_2}{E_2}$.

But strain in bar 1 = Strain in bar 2

$$\therefore \quad = \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \quad \text{...}(v)$$

From equations (*iv*) and (*v*), the stresses σ_1 and σ_2 can be determined. By substituting the values of σ_1 and σ_2 in equations (*ii*) and (*iii*), the load carried by different materials may be computed.

Modular Ratio. The ratio of $\dfrac{E_1}{E_2}$ is called the modular ratio of the first material to the second.

Lect-5
Statically indeterminate problems, Temperature stress in composite bar

Statically indeterminate problems

When a system comprises two or more members of different materials, the forces in various members cannot be determined by the principle of statics alone. Such systems are known as *statically indeterminate systems*. In such systems, additional equations are required to supplement the equations of statics to determine the unknown forces. Usually, these equations are obtained from deformation conditions of the system and are known as *compatibility equations*. A compound bar is a case of an indeterminate system and is discussed below:

Compound Bar

A bar consisting of two or more bars of different materials in parallel is known as a *composite* or *compound bar*. In such a bar, the sharing of load by each can be found by applying equilibrium and the compatibility equations.

Consider the case of a solid bar enclosed in a hollow tube as shown in Fig. 1.13. Let the subscripts 1 and 2 denote the solid bar and the hollow tube respectively.

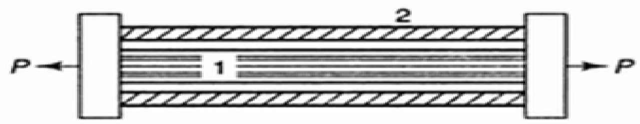


Fig. 1.13

Equilibrium equation As the total load must be equal to the load taken by individual members,

$$P = P_1 + P_2 \quad \text{(i)}$$

Compatibility equation The deformation of the bar must be equal to the tube.

$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2} \quad \text{or} \quad P_1 = \frac{P_2 A_1 E_1}{A_2 E_2} \quad \text{(ii)}$$

Inserting (ii) in (i),

$$P = \frac{P_2 A_1 E_1}{A_2 E_2} + P_2 = \frac{P_2 A_1 E_1 + P_2 A_2 E_2}{A_2 E_2} = \frac{P_2 (A_1 E_1 + A_2 E_2)}{A_2 E_2}$$

or

$$P_2 = \frac{P.A_2 E_2}{A_1 E_1 + A_2 E_2} \quad (1.12)$$

Similarly,

$$P_1 = \frac{P.A_1 E_1}{A_1 E_1 + A_2 E_2} \quad (1.13)$$

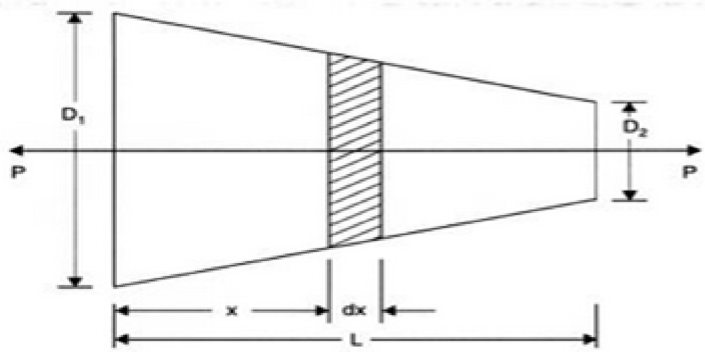
Analysis of uniformly tapering circular Bar

A bar uniformly tapering from a diameter D_1 at one end to a diameter D_2 at the other end is shown in Fig. 1.13.

Let P = Axial tensile load on the bar

L = Total length of the bar

E = Young's modulus.



Consider a small element of length dx of the bar at a distance x from the left end. Let the diameter of the bar be D_x at a distance x from the left end.

Then
$$D_x = D_1 - \left(\frac{D_1 - D_2}{L}\right) x$$
$$= D_1 - kx \qquad \text{where } k = \frac{D_1 - D_2}{L}$$

Area of cross-section of the bar at a distance x from the left end,
$$A_x = \frac{\pi}{4} D_x^2 = \frac{\pi}{4} (D_1 - k.x)^2.$$

Now the stress at a distance x from the left end is given by,
$$\sigma_x = \frac{\text{Load}}{A_x}$$
$$= \frac{P}{\frac{\pi}{4}(D_1 - k.x)^2} = \frac{4P}{\pi (D_1 - k.x)^2}.$$

The strain e_x in the small element of length dx is obtained by using equation (1.5).

$$\therefore \quad e_x = \frac{\text{Stress}}{E} = \frac{\sigma_x}{E}$$

$$= \frac{4P}{\pi (D_1 - k.x)^2} \times \frac{1}{E} = \frac{4P}{\pi E (D_1 - k.x)^2}$$

$\therefore$ Extension of the small elemental length dx

$$= \text{Strain}.\ dx = e_x\ .\ dx$$
$$= \frac{4P}{\pi E (D_1 - k\ .\ x)^2}\ .\ dx \qquad ...(i)$$

Total extension of the bar is obtained by integrating the above equation between the limits 0 and L.

$\therefore$ **Total extension,**

$$dL = \int_0^L \frac{4P \cdot dx}{\pi E(D_1 - k.x)^2} = \frac{4P}{\pi E}\int_0^L (D_1 - k.x)^{-2} \cdot dx$$

$$= \frac{4P}{\pi E}\int_0^L \frac{(D_1 - k \cdot x)^{-2} \times (-k)}{(-k)} \cdot dx \quad \text{[Multiplying and dividing by } (-k)\text{]}$$

$$= \frac{4P}{\pi E}\left[\frac{(D_1 - k \cdot x)^{-1}}{(-1)\times(-k)}\right]_0^L = \frac{4P}{\pi Ek}\left[\frac{1}{(D_1 - k \cdot x)}\right]_0^L$$

$$= \frac{4P}{\pi Ek}\left[\frac{1}{D_1 - k \cdot L} - \frac{1}{D_1 - k \times 0}\right]$$

$$= \frac{4P}{\pi Ek}\left[\frac{1}{D_1 - k \cdot L} - \frac{1}{D_1}\right]$$

Substituting the value of $k = \frac{D_1 - D_2}{L}$ **in the above equation, we get**

Total extension,

$$dL = \frac{4P}{\pi E \cdot \left(\frac{D_1 - D_2}{L}\right)}\left[\frac{1}{D_1 - \left(\frac{D_1 - D_2}{L}\right) \cdot L} - \frac{1}{D_1}\right]$$

$$= \frac{4PL}{\pi E \cdot (D_1 - D_2)}\left[\frac{1}{D_1 - D_1 + D_2} - \frac{1}{D_1}\right]$$

$$= \frac{4PL}{\pi E \cdot (D_1 - D_2)}\left[\frac{1}{D_2} - \frac{1}{D_1}\right]$$

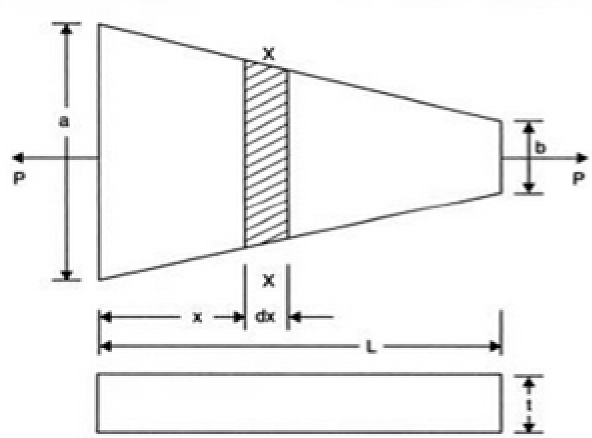
$$= \frac{4PL}{\pi E \cdot (D_1 - D_2)} \times \frac{(D_1 - D_2)}{D_1 D_2} = \frac{4PL}{\pi E D_1 D_2}$$

If the rod is of uniform diameter, then $D_1 - D_2 = D$

$\therefore$ **Total extension,** $dL = \frac{4PL}{\pi E \cdot D^2}$

Analysis of uniformly tapering rectangular bar

A bar of constant thickness and uniformly tapering in width from one end to the other end is shown in Fig. 1.14.



Let
P = Axial load on the bar
L = Length of bar
a = Width at bigger end
b = Width at smaller end
E = Young's modulus
t = Thickness of bar

Consider any section X-X at a distance x from the bigger end.

Width of the bar at the section X-X

$$= a - \frac{(a-b)x}{L}$$

$$= a - kx \qquad \text{where } k = \frac{a-b}{L}$$

Thickness of bar at section X-$X = t$

$\therefore$ Area of the section X-X

$$= \text{Width} \times \text{thickness}$$

$$= (a - kx)t$$

$\therefore$ Stress on the section X-X

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{(a-kx)t}$$

Extension of the small elemental length dx

$$= \text{Strain} \times \text{Length } dx$$

$$= \frac{\text{Stress}}{E} \times dx \qquad \left(\because \text{ Strain} = \frac{\text{Stress}}{E}\right)$$

$$= \frac{\left(\frac{P}{(a-kx)t}\right)}{E} \times dx \qquad \left(\because \text{ Stress} = \frac{P}{(a-kx)t}\right)$$

$$= \frac{P}{E(a-kx)t}\, ax$$

Total extension of the bar is obtained by integrating the above equation between the limits 0 and L.

$\therefore$ Total extension,

$$dL = \int_0^L \frac{P}{E(a-kx)t}\, dx = \frac{P}{Et}\int_0^L \frac{dx}{(a-kx)}$$

$$= \frac{P}{Et} \cdot \log_e \Big[(a-kx)\Big]_0^L \times \left(-\frac{1}{k}\right) = -\frac{P}{Etk}\,[\log_e (a-kL) - \log_e a]$$

$$= \frac{P}{Etk}\,[\log_e a - \log_e (a-kL)] = \frac{P}{Etk}\left[\log_e\left(\frac{a}{a-kL}\right)\right]$$

$$= \frac{P}{Et\left(\frac{a-b}{L}\right)}\left[\log_e\left(\frac{a}{a-\left(\frac{a-b}{L}\right)L}\right)\right] \qquad \left(\because\ k = \frac{a-b}{L}\right)$$

$$= \frac{PL}{Et(a-b)} \log_e \frac{a}{b}\ .$$

Thermal Stress

Thermal stresses are the stresses induced in a body due to change in temperature. Thermal stresses are set up in a body, when the temperature of the body is raised or lowered and the body is not allowed to expand or contract freely. But if the body is allowed to expand or contract freely, no stresses will be set up in the body.

Consider a body which is heated to a certain temperature.

Let L = Original length of the body,

T = Rise in temperature,

E = Young's Modulus,

α = Co-efficient of linear expansion,

dL = Extension of rod due to rise of temperature.

If the rod is free to expand, then extension of the rod is given by

$$dL = \alpha.\ T.L. \qquad ...(1.13)$$

This is shown in Fig. 1.23 (a) in which AB represents the original length and BB' represents the increase in length due to temperature rise. Now suppose that an external compressive load, P is applied at B' so that the rod is decreased in its length from $(L + \alpha TL)$ to L as shown in Figs. 1.23 (b) and (c).

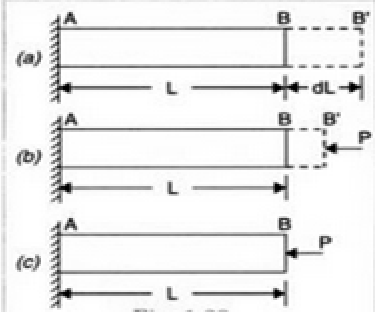


Fig. 1.23

Then compressive strain $= \dfrac{\text{Decrease in length}}{\text{Original length}}$

$$= \frac{\alpha . T . L}{L + \alpha . T . L} \approx \frac{\alpha TL}{L} = \alpha.T$$

But $\dfrac{\text{Stress}}{\text{Strain}} = E$

$\therefore$ Stress = Strain × E = $\alpha.T.E$

And load or thrust on the rod = Stress × Area = $\alpha.T.E \times A$

If the ends of the body are fixed to rigid supports, so that its expansion is prevented, then compressive stress and strain will be set up in the rod. These stresses and strains are known as thermal stresses and thermal strain.

$\therefore$ Thermal strain, $e = \dfrac{\text{Extension prevented}}{\text{Original length}}$

$$= \frac{dL}{L} = \frac{\alpha . T . L}{L} = \alpha.T$$

And thermal stress, σ = Thermal strain × E

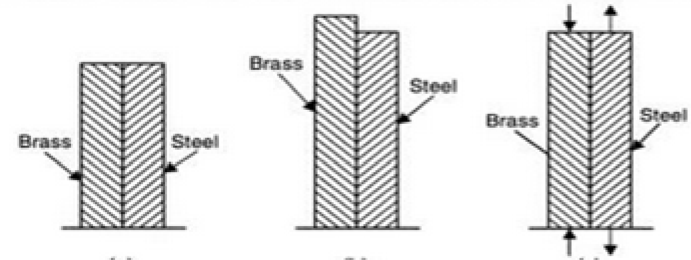
$= \alpha.T.E.$

Thermal stress is also known as temperature stress.

And thermal strain is also known as temperature strain.

Temperature stress in composite bar

Fig. 1.24 (a) shows a composite bar consisting of two members, a bar of brass and another of steel. Let the composite bar be heated through some temperature. If the members are free to expand then no stresses will be induced in the members. But the two members are rigidly fixed and hence the composite bar as a whole will expand by the same amount. As the co-efficient of linear expansion of brass is more than that of the steel, the brass will expand more than the steel. Hence the free expansion of brass will be more than that of the steel. But both the members are not free to expand, and hence the expansion of the composite bar, as a whole, will be less than that of the brass, but more than that of the steel. Hence the stress



induced in the brass will be compressive whereas the stress in steel will be tensile as shown in Fig. 1.24 (*c*). Hence the load or force on the brass will be compressive whereas on the steel the load will be tensile.

Let A_b = Area of cross-section of brass bar

σ_b = Stress in brass

e_b = Strain in brass

α_b = Co-efficient of linear expansion for brass

E_b = Young's modulus for copper

A_s, σ_s, e_s and α_s = Corresponding values of area, stress, strain and co-efficient of linear expansion for steel, and

E_s = Young's modulus for steel.

δ = Actual expansion of the composite bar

Now load on the brass = Stress in brass × Area of brass

$= \sigma_b \times A_b$

And load on the steel $= \sigma_s \times A_s$

For the equilibrium of the system, compression in copper should be equal to tension in the steel

or Load on the brass = Load on the steel

$\therefore \quad \sigma_b \times A_b = \sigma_s \times A_s.$

Also we know that actual expansion of steel

= Actual expansion of brass ...(*i*)

But actual expansion of steel

= Free expansion of steel + Expansion due to tensile stress in steel

$$= \alpha_s \, . \, T \, . \, L + \frac{\sigma_s}{E_s} \, . \, L$$

And actual expansion of copper

= Free expansion of copper – Contraction due to compressive stress induced in brass

$$= \alpha_b \, . \, T \, . \, L - \frac{\sigma_b}{E_b} . \, L$$

Substituting these values in equation (*i*), we get

$$\alpha_s \times T \times L + \frac{\sigma_s}{E_s} \times L = \alpha_b \times T \times L - \frac{\sigma_b}{E_b} \times L$$

or
$$\alpha_s T + \frac{\sigma_s}{E_s} = \alpha_b \times T - \frac{\sigma_b}{E_b}$$

where T = Rise of temperature.

Lect-6
Complimentary shear stress, Shear strain, Modulus of rigidity, Poisson's ratio, Bulk Modulus, Relationship between elastic constants.

Complimentary shear stress

It states that a set of shear stresses across a plane is always accompanied by a set of balancing shear stresses (*i.e.*, of the same intensity) across the plane and normal to it.

Proof. Fig. 2.8 shows a rectangular block *ABCD*, subjected to a set of shear stresses of intensity τ on the faces *AB* and *CD*. Let the thickness of the block normal to the plane of the paper is unity.

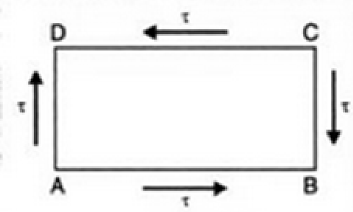


Fig. 2.8

The force acting on face *AB*

$= \text{Stress} \times \text{Area}$

$= \tau \times AB \times 1 = \tau . AB$

Similarly force acting on face *CD*

$= \tau \times CD \times 1 = \tau . CD$

$= \tau . AB$ $\quad (\because \; CD = AB)$

The forces acting on the faces *AB* and *CD* are equal and opposite and hence these forces will form a couple.

The moment of this couple $= \text{Force} \times \text{Perpendicular distance}$

$= \tau . AB \times AD$ $\quad ...(i)$

If the block is in equilibrium, there must be a restoring couple whose moment must be equal to the moment given by equation (*i*). Let the shear stress of intensity τ' is set up on the faces *AD* and *CB*.

The force acting on face $AD = \tau' \times AD \times 1 = \tau' . AD$

The force acting on face $BC = \tau' \times BC \times 1 = \tau' BC = \tau' . AD$ $\quad (\because \; BC = AD)$

As the force acting on faces *AD* and *BC* are equal and opposite, these forces also forms a couple.

Moment of this couple = Force × Distance $= \tau' . AD \times AB$ $\quad ...(ii)$

For the equilibrium of the block, the moments of couples given by equations (*i*) and (*ii*) should be equal

The stress τ' is known as complementary shear and the two stresses (τ and τ') at right angles together constitute a state of simple shear. The direction of the shear stresses on the block are either both towards or both away from a corner.

<u>Relationship between stress and strain</u>

1.9.1. For One-Dimensional Stress System. The relationship between stress and strain for a unidirectional stress (*i.e.*, for normal stress in one direction only) is given by **Hooke's law,** which states that when a material is loaded within its elastic limit, the normal stress developed is proportional to the strain produced. This means that the ratio of the normal stress to the corresponding strain is a constant within the elastic limit. This constant is represented by E and is known as modulus of elasticity or Young's modulus of elasticity.

$$\therefore \quad \frac{\text{Normal stress}}{\text{Corresponding strain}} = \text{Constant} \quad \text{or} \quad \frac{\sigma}{e} = E$$

where σ = Normal stress, e = Strain and E = Young's modulus

or $$e = \frac{\sigma}{E} \qquad \text{...[1.7 (A)]}$$

The above equation gives the stress and strain relation for the normal stress in one direction.

1.9.2. For Two-Dimensional Stress System. Before knowing the relationship between stress and strain for two-dimensional stress system, we shall have to define longitudinal strain, lateral strain, and Poisson's ratio.

1. Longitudinal strain. When a body is subjected to an axial tensile load, there is an increase in the length of the body. But at the same time there is a decrease in other dimensions of the body at right angles to the line of action of the applied load. Thus the body is having axial deformation and also deformation at right angles to the line of action of the applied load (*i.e.*, lateral deformation).

The ratio of axial deformation to the original length of the body is known as longitudinal (or linear) strain. The longitudinal strain is also defined as the deformation of the body per unit length in the direction of the applied load.

Let L = Length of the body,

P = Tensile force acting on the body,

δL = Increase in the length of the body in the direction of P.

Then, longitudinal strain = $\frac{\delta L}{L}$.

2. Lateral strain. The strain at right angles to the direction of applied load is known as lateral strain. Let a rectangular bar of length L, breadth b and depth d is subjected to an axial tensile load P as shown in Fig. 1.5. The length of the bar will increase while the breadth and depth will decrease.

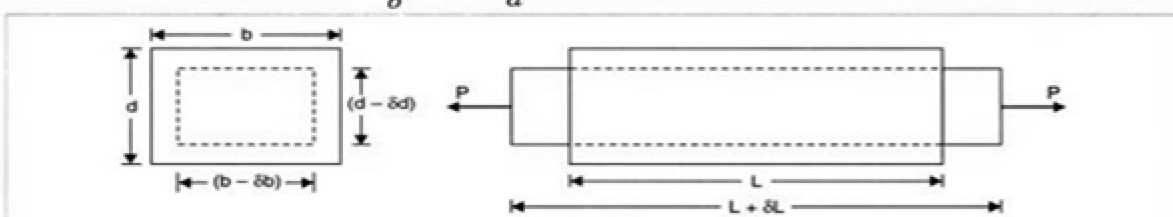
Let δL = Increase in length,

δb = Decrease in breadth, and

δd = Decrease in depth.

Then longitudinal strain = $\frac{\delta L}{L}$ ...[1.7 (B)]

and lateral strain = $\frac{\delta b}{b}$ or $\frac{\delta d}{d}$ ...[1.7 (C)]



3. Poisson's ratio. The ratio of lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called **Poisson's ratio** and it is generally denoted by μ. Hence mathematically,

$$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} \qquad \text{...[1.7 (D)]}$$

or Lateral strain = $\mu \times$ Longitudinal strain

As lateral strain is opposite in sign to longitudinal strain, hence algebraically, lateral strain is written as

$$\text{Lateral strain} = -\mu \times \text{Longitudinal strain} \qquad \text{...[1.7 (E)]}$$

4. Relationship between stress and strain. Consider a two-dimensional figure *ABCD*, subjected to two mutually perpendicular stresses σ_1 and σ_2.

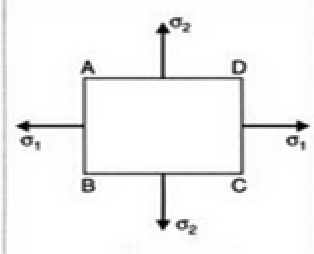


Fig. 1.5 (*a*)

Refer to Fig. 1.5 (*a*).

Let σ_1 = Normal stress in *x*-direction

σ_2 = Normal stress in *y*-direction

Consider the strain produced by σ_1.

The stress σ_1 will produce strain in the direction of *x* and also in the direction of *y*. The strain in the direction of *x* will be longitudinal strain and will be equal to $\frac{\sigma_1}{E}$ whereas the strain in the direction of *y* will be lateral strain and will be equal to $-\mu \times \frac{\sigma_1}{E}$

($\because$ Lateral strain. = $-\mu \times$ longitudinal strain)

Now consider the strain produced by σ_2.

The stress σ_2 will produce strain in the direction of *y* and also in the direction of *x*. The strain in the direction of *y* will be longitudinal strain and will be equal to $\frac{\sigma_2}{E}$ whereas the strain in the direction of *x* will be lateral strain and will be equal to $-\mu \times \frac{\sigma_2}{E}$.

Let e_1 = Total strain in *x*-direction

e_2 = Total strain in *y*-direction

Now total strain in the direction of *x* due to stresses σ_1 and $\sigma_2 = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E}$

Similarly total strain in the direction of *y* due to stresses σ_1 and $\sigma_2 = \frac{\sigma_2}{E} - \mu \frac{\sigma_1}{E}$

$\therefore$ $$e_1 = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} \quad \text{...[1.7 (F)]}$$

$$e_2 = \frac{\sigma_2}{E} - \mu \frac{\sigma_1}{E} \quad \text{...[1.7 (G)]}$$

Modulus of Rigidity or Shear Modulus. The ratio of shear stress to the corresponding shear strain within the elastic limit, is known as Modulus of Rigidity or Shear Modulus. This is denoted by *C* or *G* or *N*.

$$C \text{ (or } G \text{ or } N) = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\phi}$$

Bulk modulus

It is the ratio of direct stess to volumetric strain,it denoted by letter K

K= direct stress

RELATION BETWEEN ENGINEERING CONSTANTS

Consider a square element *ABCD* under the action of a simple shear stress τ (Fig.1.60a). The resultant distortion of the element is shown in Fig. 1.60b The total change in the corner angles is $\pm\varphi$. However, for convenience sake, the side *AB* may be considered to be fixed as shown in Fig. 1.60c. As angle φ is extremely small, *CC'* and *DD'* can be assumed to be arcs. Let *CE* be a perpendicular on the diagonal *AC'*

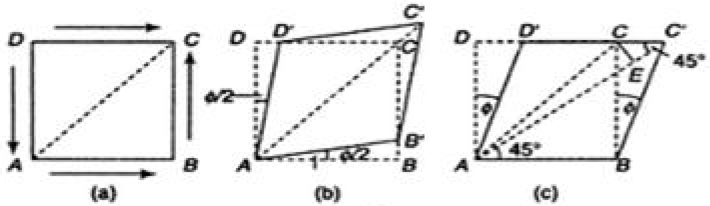


Fig. 1.60

Linear strain of the diagonal *AC* can approximately be taken as

$$\varepsilon = \frac{AC' - AC}{AC}$$

$$= \frac{EC'}{AC}$$

$$= \frac{CC'\cos 45^\circ}{AB/\cos 45^\circ}$$

$$= \frac{\varphi \cdot BC\cos^2 45^\circ}{BC} \qquad (CC' = \varphi \cdot BC \text{ and } AB = BC)$$

$$= \frac{\varphi}{2}$$

But modulus of rigidity, $G = \tau/\varphi$ or $\varphi = \tau/G$ (Eq.1.5)

$$\therefore \quad \varepsilon = \frac{\tau}{2G} \qquad \text{(i)}$$

It will be shown in Section 2.1 that in a state of simple shear on two perpendicular planes, the planes at 45° are subjected to a tensile stress (magnitude equal to that of the shear stress) while the planes at 135° are subjected to a compressive stress of the same magnitude with no shear stress on these planes. Thus, planes *AC* and *BD* are subjected to tensile and compressive stresses respectively each equal to τ in magnitude as shown in Fig. 1.61.

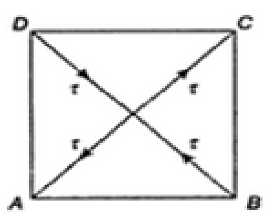


Fig. 1.61

Hence linear strain of diagonal *AC* is

$$\varepsilon = \frac{\tau}{E} - \left(-\frac{v\tau}{E}\right) = \frac{\tau}{E}(1 + v) \qquad \text{(ii)}$$

From (i) and (ii),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1 + v)$$

or $\quad E = 2G(1 + v)$

As $\quad E = 3K(1 - 2v)$

$\therefore \quad E = 2G(1 + v) = 3K(1 - 2v)$

This equation relates the elastic constants.

Also from above, $1 + v = \frac{E}{2G}$, $\therefore 2 + 2v = \frac{E}{G}$

and $\quad 1 - 2v = \frac{E}{3K}$

Adding (i) and (ii), $3 = E\left(\frac{1}{G} + \frac{1}{3K}\right) = \frac{E}{3KG}(3K + G)$

or $\quad E = \frac{9KG}{3K + G}$

Lect-7
Analysis of Biaxial Stress. Plane stress, Principal stress, Principal plane and related problem

Introduction

While analysing a stress system, the general conventions have been taken as follows:

- A tensile stress is positive and compressive stress, negative.
- A pair of shear stresses on parallel planes forming a clockwise couple is positive and a pair with counter-clockwise couple, negative.
- Clockwise angle is taken as positive and counter-clockwise, negative.

The following cases are being considered:

(i) Direct stress condition
(ii) Bi-axial stress condition
(iii) Pure shear stress condition
(iv) Bi-axial and shear stresses condition

(i) Direct Stress Condition

Let a bar be acted upon by an external force P resulting in a direct tensile stress σ_x along its length (Fig. 2.1a). The stress on any transverse section such as $BCDE$ will have a pure normal stress acting on it. The stress acting on an arbitrary plane $ACDF$ inclined at an angle θ with the vertical plane will have two components:

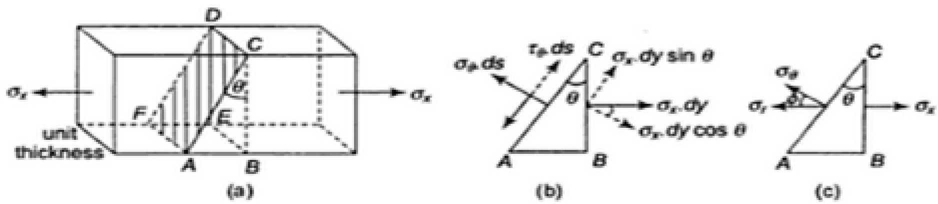


Fig. 2.1

— normal component known as *direct stress component*
— tangential component known as *shear stress component*.

These stress components can be determined from the consideration of force balance.

If the bar is imagined cut through the section $ACDF$, each portion of the bar is also in equilibrium under the action of forces due to the stresses developed. For convenience, a triangular prismatic element $ABCDEF$ containing the plane $ACDF$ can be taken for the force analysis.

Figure 2.1b shows the forces acting on the triangular element.

Let

dy = the length of the side BC
ds = the length of the side AC
σ_x = normal stress acting on the plane $BCDE$
σ_θ = normal stress acting on the plane $ACDF$
τ_θ = tangential or shear stress acting on the plane $ACDF$

Assume a unit thickness of the prism and equate the forces along normal and tangential directions to the plane $ACDF$ of the prism for its equilibrium, i.e.,

$$\sigma_\theta \cdot ds - \sigma_x.dy.\cos\theta = 0$$

or
$$\sigma_\theta = \frac{\sigma_x\, dy \cos\theta}{ds} = \frac{\sigma_x\, dy\cos\theta}{dy/\cos\theta} = \sigma_x \cos^2\theta \quad (2.1)$$

and
$$\tau_\theta \cdot ds + \sigma_x .dy.\sin\theta = 0 \quad \text{(assuming } \tau_\theta \text{ clockwise as positive)}$$

or
$$\tau_\theta = -\frac{\sigma_x\, dy\sin\theta}{ds} = -\frac{\sigma_x\, dy\sin\theta}{dy/\cos\theta} = -\sigma_x \sin\theta\cos\theta = -\frac{1}{2}\sigma_x \sin 2\theta \quad (2.2)$$

The negative sign shows that τ_θ is counter-clockwise and not clockwise on the inclined plane.

- When $\theta = 0°$, $\sigma_\theta = \sigma_x$ and $\tau_\theta = 0$
- When $\theta = 45°$, $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = -\sigma_x/2$ (maximum, counter-clockwise)
- When $\theta = 90°$, $\sigma_\theta = 0$ and $\tau_\theta = 0$
- When $\theta = 135°$ $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = \sigma_x/2$ (maximum, clockwise)

- The maximum shear stress is equal to one half the applied stress.

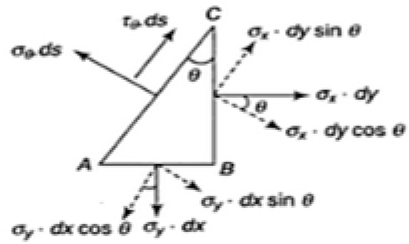
The resultant stress on the plane *ACDF* (Fig. 2.1c),

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$
$$= \sigma_x \sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}$$
$$= \sigma_x \cos\theta \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= \sigma_x \cos\theta$$

Inclination with the normal stress, $\tan\varphi = \dfrac{\sigma_x \sin\theta \cos\theta}{\sigma_x \cos^2\theta} = \tan\theta$

or $\varphi = \theta$

i.e., it is always in the direction of the applied stress.



(ii) Bi-axial Stress Condition

Let an element of a body be acted upon by two tensile stresses acting on two perpendicular planes of the body as shown in Fig. 2.3. Let *dx*, *dy* and *ds* be the lengths of the sides *AB*, *BC* and *AC* respectively. Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta \cdot ds - \sigma_x \cdot dy \cdot \cos\theta - \sigma_y \cdot dx \cdot \sin\theta = 0$$

or

$$\sigma_\theta = \frac{\sigma_x\, dy\cos\theta}{ds} + \frac{\sigma_y\, dx\sin\theta}{ds} = \frac{\sigma_x\, dy\cos\theta}{dy/\cos\theta} + \frac{\sigma_y\, dx\sin\theta}{dx/\sin\theta}$$
$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

The expression may be put in the following form,

$$\sigma_\theta = \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) = \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta \cdot \mathrm{ds} + \sigma_x .\mathrm{dy} \cdot \sin\theta - \sigma_y .\mathrm{dx} \cdot \cos\theta = 0$$

or

$$\tau_\theta = -\frac{\sigma_x\, dy \sin\theta}{ds} + \frac{\sigma_y\, dx\cos\theta}{ds} = -\frac{\sigma_x\, dy \sin}{dy/\cos\theta} + \frac{\sigma_y\, dx \cos\theta}{dx/\sin\theta}$$
$$= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta$$
$$= -(\sigma_x - \sigma_y)\sin\theta\cos\theta = -\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta$$

which indicates that it is counter-clockwise if σ_x is more than σ_y.

Resultant stress,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$
$$= \left[\left\{\frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta\right\}^2 + \left\{-\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta\right\}^2\right]^{1/2}$$
$$= \left[\frac{1}{4}(\sigma_x+\sigma_y)^2 + \frac{1}{4}(\sigma_x-\sigma_y)^2\cos^2 2\theta + \frac{1}{2}(\sigma_x+\sigma_y)(\sigma_x-\sigma_y)\cos 2\theta + \frac{1}{4}(\sigma_x-\sigma_y)^2\sin^2 2\theta\right]^{1/2}$$
$$= \left[\frac{1}{4}(\sigma_x+\sigma_y)^2 + \frac{1}{4}(\sigma_x-\sigma_y)^2 + \frac{1}{2}(\sigma_x^2-\sigma_y^2)\cos 2\theta\right]^{1/2}$$
$$= \left[\frac{1}{4}(\sigma_x^2+\sigma_y^2+2\sigma_x\sigma_y+\sigma_x^2+\sigma_y^2-2\sigma_x\sigma_y) + \frac{1}{2}(\sigma_x^2-\sigma_y^2)\cos 2\theta\right]^{1/2}$$
$$= \left[\frac{1}{2}(\sigma_x^2+\sigma_y^2) + \frac{1}{2}(\sigma_x^2-\sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2(1+\cos 2\theta) + \frac{1}{2}\sigma_y^2(1-\cos 2\theta)\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2 \cdot 2\cos^2\theta + \frac{1}{2}\sigma_y^2 \cdot 2\sin^2\theta\right]^{1/2}$$

$$= \sqrt{\sigma_x^2\cos^2\theta + \sigma_y^2\sin^2\theta}$$

and the angle of inclination of the resultant with σ_θ,

$$\tan\varphi = \frac{\tau_\theta}{\sigma_\theta} = \frac{-(\sigma_x - \sigma_y)\sin\theta\cos\theta}{\sigma_x\cos^2\theta + \sigma_y\sin^2\theta} = \frac{\sigma_y - \sigma_x}{\sigma_x\cot\theta + \sigma_y\tan\theta}$$

For greatest obliquity or inclination of the resultant with the normal stress,

$$\frac{d(\tan\varphi)}{d\theta} = 0$$

or

$$-\sigma_x\operatorname{cosec}^2\theta + \sigma_y\sec^2\theta = 0 \quad \text{or} \quad \sigma_x\operatorname{cosec}^2\theta = \sigma_y\sec^2\theta$$

$$\tan^2\theta = \frac{\sigma_x}{\sigma_y} \quad \text{or} \quad \tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}} \tag{2.10}$$

∴

$$\tan\varphi_{\max} = \frac{\sigma_y - \sigma_x}{\sigma_x\sqrt{\sigma_y/\sigma_x} + \sigma_y\sqrt{\sigma_x/\sigma_y}} = \frac{\sigma_y - \sigma_x}{2\sqrt{\sigma_x\sigma_y}} \tag{2.10a}$$

The angle of inclination of the resultant with σ_x,

$$\tan\alpha = \frac{\sigma_y \cdot dx}{\sigma_x \cdot dy} = \frac{\sigma_y \cdot dy \cdot \tan\theta}{\sigma_x \cdot dy} = \frac{\sigma_y}{\sigma_x}\tan\theta \tag{2.11}$$

The above results show the following:

- The normal stress on the inclined plane varies between the values of σ_x and σ_y as the angle θ is increased from 0° to 90°. For equal values of the two axial stresses ($\sigma_x = \sigma_y$), σ_θ is always equal to σ_x or σ_y.
- The shear stress is zero on planes with angles 0° and 90°, i.e., on horizontal and vertical planes. It has maximum value numerically equal to one half the difference between given normal stresses which occurs on planes at ± 45° to the given planes.

$$\tau_{\max} = \pm\frac{1}{2}(\sigma_x - \sigma_y) \tag{2.12}$$

and the normal stress across the same plane,

and the normal stress across the same plane,

$$\sigma_{45^\circ} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 90^\circ = \frac{1}{2}(\sigma_x + \sigma_y) \qquad (2.1$$

(iii) Pure Shear Stress Condition

Let an element of a body be acted upon by shear stresses on its two perpendicular faces as shown in Fig. 2.4.
Let dx, dy and ds be the lengths of the sides AB, BC and AC respectively.
Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta \cdot ds - \tau \cdot dx \cdot \cos\theta - \tau \cdot dy \cdot \sin\theta = 0$$

or

$$\sigma_\theta = \frac{\tau\, dx\cos\theta}{ds} + \frac{\tau\, dy\sin\theta}{ds} = \frac{\tau\, dx\cos\theta}{dx/\sin\theta} + \frac{\tau\, dy\sin\theta}{dy/\cos\theta}$$

$$= \tau\sin\theta\cos\theta + \tau\sin\theta\cos\theta = \tau\cdot\sin 2\theta \tag{2.15}$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta \cdot \mathrm{ds} - \tau \cdot \mathrm{dy} \cdot \cos\theta + \tau \cdot \mathrm{dx} \cdot \sin\theta = 0$$

or

$$\tau_\theta = \frac{\tau\, dy\cos\theta}{ds} - \frac{\tau\, dx\sin\theta}{ds} = \frac{\tau\, dy\cos\theta}{dy/\cos\theta} - \frac{\tau\, dx\sin}{dx/\sin\theta}$$

$$= \tau\cos^2\theta - \tau\sin^2\theta = \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right] = \tau\cos 2\theta \tag{2.16}$$

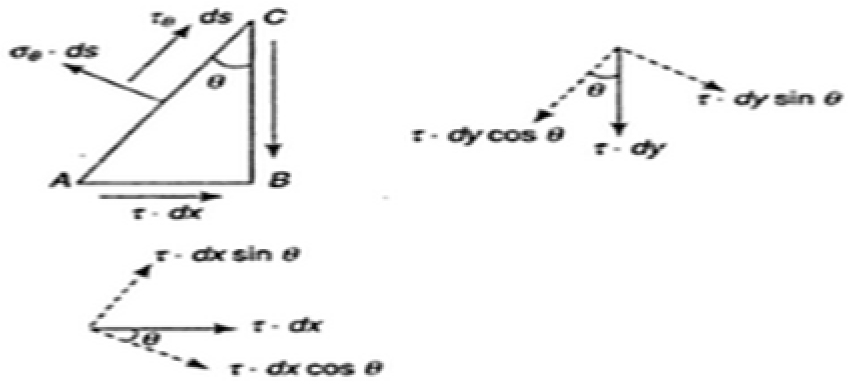


Fig. 2-4

Fig. 2.4

which shows that it is up the plane for $\theta < 45°$ and down the plane for $\theta > 45°$.

The resultant stress on the plane AC, $\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \tau\sqrt{(\sin 2\theta)^2 + (\cos 2\theta)^2}$

$$= \tau$$

Inclination with the direction of shear stress planes, $\tan\varphi = \dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

or $$\varphi = 2\theta$$

(iv) Bi-axial and Shear Stresses Condition

Let an element of a body be acted upon by two tensile stresses along with shear stresses acting on two perpendicular planes of the body as shown in Fig. 2.5. Let dx, dy and ds be the lengths of the sides AB, BC and AC respectively.

Considering unit thickness of the body and resolving the forces in the direction of σ_φ,

$$\sigma_\theta \cdot ds = \sigma_x \cdot dy \cdot \cos\theta + \sigma_y \cdot dx \cdot \sin\theta + \tau \cdot dy \cdot \sin\theta + \tau \cdot dx \cdot \cos\theta$$

or
$$\sigma_\theta = \frac{\sigma_x\, dy \cos\theta}{ds} + \frac{\sigma_y\, dx \sin\theta}{ds} + \frac{\tau \cdot dy \sin\theta}{ds} + \frac{\tau \cdot dx \cos\theta}{ds}$$

$$= \frac{\sigma_x\, dy \cos\theta}{dy/\cos\theta} + \frac{\sigma_y\, dx \sin\theta}{dx/\sin\theta} + \frac{\tau\, dy \sin\theta}{dy/\cos\theta} + \frac{\tau\, dx \cos\theta}{dx/\sin\theta}$$

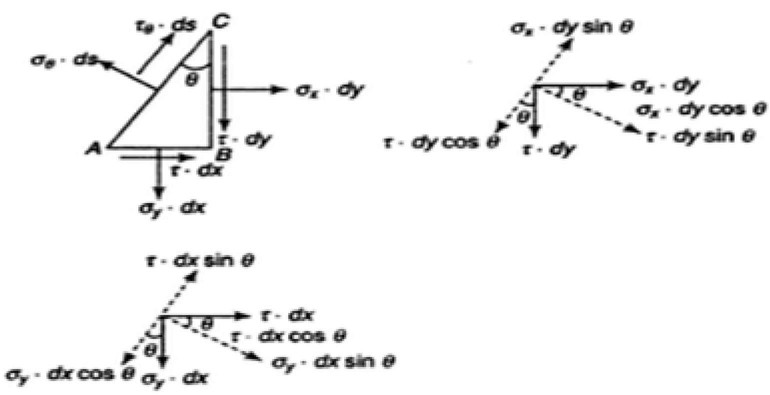


Fig. 2.5

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin\theta\cos\theta + \tau \sin\theta\cos\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$

$$= \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) + \tau \cdot \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

Resolving the forces in the direction of τ,

$$\tau\theta \cdot ds + \sigma_x \cdot dy \cdot \sin\theta - \sigma_y \cdot dx \cdot \cos\theta - \tau \cdot dy \cdot \cos\theta + \tau \cdot dx \cdot \sin\theta = 0$$

or

$$\tau_\theta = -\frac{\sigma_x dy \sin\theta}{ds} + \frac{\sigma_y dx \cos\theta}{ds} + \frac{\tau \cdot dy \cos\theta}{ds} - \frac{\tau \cdot dx \sin\theta}{ds} -$$

$$= -\frac{\sigma_x dy \sin}{dy/\cos\theta} + \frac{\sigma_y dx \cos\theta}{dx/\sin\theta} + \frac{\tau dy \cos\theta}{dy/\cos\theta} - \frac{\tau dx \sin}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau\cos^2\theta - \tau\sin^2\theta$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right]$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta \qquad (2.21)$$

Equations 2.19, 2.20 and 2.21 can be used to determine the stresses on any inclined plane in a material under a general state of stress.

To determine the planes having maximum and minimum values of direct stress, differentiate Eq. 2.20 with respect to θ and equate to zero, i.e.,

$$\frac{d\sigma_\theta}{d\theta} = 0 - \frac{1}{2}(\sigma_x - \sigma_y)\, 2\sin 2\theta + 2\tau \cdot \cos 2\theta = 0$$

or

$$\frac{1}{2}(\sigma_x - \sigma_y) 2\sin 2\theta = 2\tau \cdot \cos 2\theta$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$$

Lect-8

Principal stress, Principal plane ,Mohr's Circle for Biaxial Stress.and related problem

Principal stress, Principal plane

In general, a body may be acted upon by direct stresses and shear stresses. However, it will be seen that even in such complex systems of loading, there exist three mutually perpendicular planes, on each of which the resultant stress is wholly normal. These are known as *principal planes* and the normal stress across these planes, as *principal stresses*. The larger of the two stresses σ_1 is called the *major principal stress*, and the smaller one σ_2, as the *minor principal stress*. The corresponding planes are known as *major* and *minor principal planes*. In two-dimensional problems, the third principal stress is taken to be zero.

As shear stress is zero in principal planes,

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta = 0 \qquad \text{(Eq. 2.21)}$$

or

$$\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau\cos 2\theta$$

or

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \qquad (2.23)$$

which provides two values of 2θ differing by 180° or two values of θ differing by 90°. Thus, the two principal planes are perpendicular to each other (Also, refer Eq. 2.22).

From Fig. 2.6,

$$\sin 2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\cos 2\theta = \pm\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

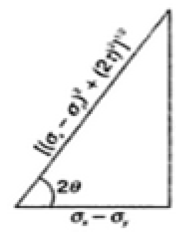


Fig. 2.6

Right-hand sides of both the above equations should have the same signs, positive or negative while using them. Substituting these values of sin2θ and cos2θ in Eq. 2.20, two values of the direct stresses, i.e., of principal stresses corresponding to two values of 2θ are obtained.

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau\cdot\sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}} \pm \tau\cdot\frac{2\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} \qquad (2.24)$$

Maximum shear stress-

In any complex system of loading, the maximum and the minimum normal stresses are the principal stresses and the shear stress is zero in their planes. To find the maximum value of shear stress and its plane in such a system, consider the equation for shear stress in a plane, i.e.,

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta \qquad \text{(Eq. 2.21)}$$

For maximum value of τ_θ, differentiate it with respect to θ and equate to zero,

$$\frac{d\tau_\theta}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau\sin 2\theta = 0$$

or

$$\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau} \qquad (2.25)$$

This indicates that there are two values of 2θ differing by 180° or two values θ differing by 90°. Thus, maximum shear stress planes lie at right angle to each other.

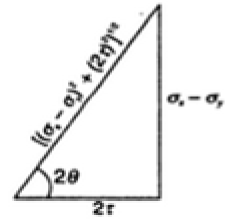


Fig. 2.7

Now, as $\tan 2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau}$ can be represented as shown in Fig. 2.7,

$$\sin 2\theta = \mp \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}; \qquad \cos 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}}$$

Right-hand sides of both the above equations should have the opposite signs, if one positive the other negative while using them. Substituting these values of $\sin 2\theta$ and $\cos 2\theta$ in Eq. 2.21, two values of the shear stress are obtained.

$$\therefore \qquad \tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

As maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

and minimum principal stress, $\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

Subtracting (ii) from (i), $\sigma_1 - \sigma_2 = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$\therefore \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$

Thus, in general, $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

Now, principal planes are given by, $\tan 2\theta_p = \frac{2\tau}{\sigma_x - \sigma_y}$

and planes of maximum shear stress, $\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau}$

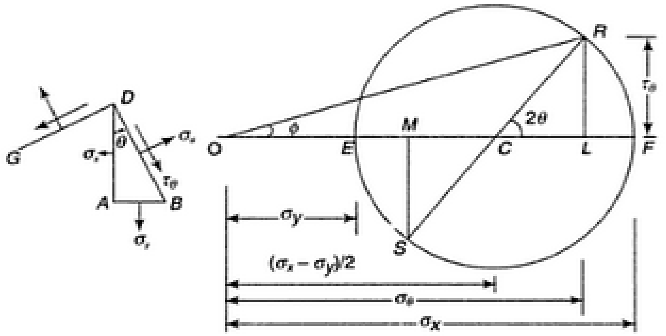
Multiplying the two, $\tan 2\theta_p \cdot \tan 2\theta_s = -1$ which means $2\theta_s = 2\theta_p + 90°$ or $\theta_s = \theta_p + 45°$

MOHR'S CIRCLE-

The stress components on any inclined plane can easily be found with the help of a geometrical constructic known as *Mohr's stress circle*.

Two Perpendicular Direct Stresses

Let the material of a body at a point be subjected to two like direct tensile stresses σ_x and σ_y ($\sigma_x > \sigma_y$), on tw perpendicular planes *AD* and *AB* respectively (Fig. 2.8).



Make the following constructions:

- On x-axis, take $OF = \sigma_x$ and $OE = \sigma_y$ to some scale. A stress is taken towards the right of the origin O (positive) if tensile and toward left (negative) if compressive.
- Bisect EF at C.
- With C as centre and CE (= CF) as radius, draw a circle.

  The radius CF represents the plane AD (of direct stress σ_x) and CE, the plane AB (of direct stress σ_y). Note that the two planes AD and AB which are at 90° are represented at 180° apart (or at double the angle) in the Mohr's circle. This indicates that any angular position of a plane can be located at double the angle from a particular plane.
- Locate an inclined plane in this circle by marking a radial line at double the angle at which the required plane is inclined with a given plane, e.g., if the plane BD is inclined at angle θ with plane AD in the counter-clockwise direction, then mark radius CR at an angle 2θ with CF in the counter-clockwise direction.
- Draw $LR \perp x$-axis. Join OR.

Now, it can be shown that OL and LR represent the normal and the shear stress components on the inclined plane BD.

From the geometry of the figure,

$$OC = \frac{1}{2}(OC + OC) = \frac{1}{2}(OF - CF) + (OE + CE)$$

$$= \frac{1}{2}(OF - CF) + (OE + CF) \qquad \text{......}(CE = CF)$$

$$= \frac{1}{2}(OF + OE) = \frac{1}{2}(\sigma_x + \sigma_y)$$

$$CL = CR \cos 2\theta = CF \cos 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta \qquad (CR = CF)$$

Thus $OL = OC + CL = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta = \sigma_\theta$ ...(Refer Eq.2.23)

And $LR = CR \sin 2\theta = CF \sin 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta = \tau_\theta$ ...(Refer Eq.2.24)

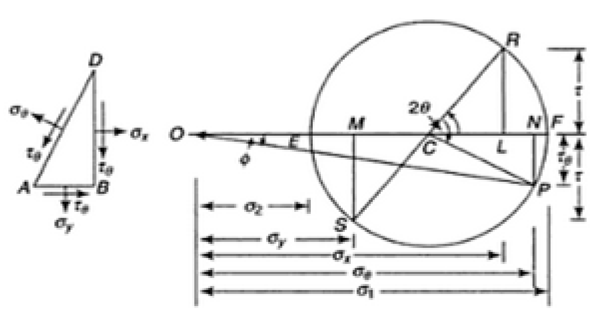
Two Perpendicular Direct Stresses with Simple Shear

In the above-discussed case, CR and CS represent two perpendicular planes having direct tensile stresses OL and OM and shear stresses LR (clockwise) and MS (= LR, counter-clockwise) respectively. Now, if these happen to be the known stresses on two perpendicular planes, then stresses on any other inclined plane can easily be found by locating that plane relative to any of these planes.

Let CR and CS represent two perpendicular planes BD and AB respectively so that $OL = \sigma_x$, $OM = \sigma_y$ and LR and MS each equal to τ in the clockwise and counter-clockwise directions respectively (Fig. 2.9). Now if it is desired to find stresses on an inclined plane at angle θ clockwise with plane BD, a radial line CP may be drawn at angle 2θ in the clockwise direction with CR. Then ON and NP will represent the direct and shear components respectively on the plane AD and the resultant is given by OP.

Thus the procedure may be summarised as follows:

- Take OL and OM as the direct components of the two perpendicular stresses σ_x and σ_y.
- At L and M draw ⊥s LR and MS on the x-axis each equal to τ using the same scale as for the direct stresses. For the stress system shown in Fig. 2.8, LR is taken upwards as the direction on plane BD is clockwise and MS downwards as the direction on plane AB is counter-clockwise.
- Bisect LM at C and draw a circle with C as centre and radius equal to CR (= CS).



- Rotate the radial line *CR* through angle 2θ in the clockwise direction if θ is taken clockwise and let it take the position *CP*.
- Draw $NP \perp$ on *x*-axis. Join *OP*.

It can be proved that *ON* and *NP* represent the normal and the shear stress components on the inclined plane *AD*.

From the geometry of the figure,

$$OC = \frac{1}{2}(\sigma_x + \sigma_y) \qquad \text{as before.}$$

$$\begin{aligned} CN &= CP\cos(2\theta - \beta) \\ &= CR\cos(2\theta - \beta) && \ldots(CP = CR) \\ &= CR(\cos 2\theta \cos\beta + \sin 2\theta \sin\beta) \\ &= (CR\cos\beta)\cos 2\theta + (CR\sin\beta)\sin 2\theta \\ &= CL\cos 2\theta + LR\sin 2\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau\cdot\sin 2\theta && \ldots(CL = OL - OM) \end{aligned}$$

Thus $ON = OC + CN = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau\cdot\sin 2\theta = \sigma_\theta$ …(Eq.2.20)

and

$$\begin{aligned} NP &= CP\sin(2\theta - \beta) \\ &= CR\sin(2\theta - \beta) \\ &= CR(\sin 2\theta\cos\beta - \cos 2\theta\sin\beta) \\ &= (CR\cos\beta)\sin 2\theta - (CR\sin\beta)\cos 2\theta \\ &= CL\sin 2\theta - LR\cos 2\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta = -\tau_\theta && \ldots(\text{Eq.2.21}) \end{aligned}$$

As *NP* is below the *x*-axis, therefore, the shear stress is negative or counter-clockwise.

Mathematically, $NP = -\left[\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta\right] = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta]$

Principal Stresses

As shear stress is zero on the principal planes, *OF* represents the major principal plane with maximum normal stress. In a similar way, *OE* represents the minor principal plane.

$$\begin{aligned} OF = OC + CF = OC + CR &= OC + \sqrt{CL^2 + LR^2} \\ &= \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + \tau^2} \\ &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} \\ &= \text{Major principal stress} \end{aligned}$$

$$\begin{aligned} OE = OC - CE = OC - CR &= OC - \sqrt{CL^2 + LR^2} \\ &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} \\ &= \text{Minor principal stress} \end{aligned}$$

Lect-9
Two dimensional state of strain

If direct and shear strains along x- and y-directions are known, normal strain (ε_θ) and the shear strain (φ_θ) in a direction at angle θ with the x-direction of a body can be found by the following method:

Normal Strain

Let a rectangular element $OACB$ with angle of the diagonal θ with the direction of ε_x or x-axis distorts to become a parallelogram $OA'C'B'$ under the action of linear strains ε_x, ε_y and shear strain φ as shown in Fig. 2.36. Point C moves to C'. Let r be the length of the diagonal OC.

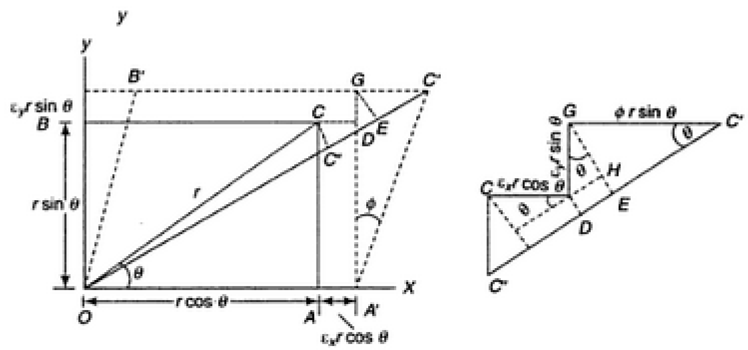


Fig. 2.36

Now, elongation of the diagonal $= C''C' = C''D + DE + EC'$

$$= (\varepsilon_x \cdot r \cdot \cos\theta)\cos\theta + (\varepsilon_y \cdot r \cdot \sin\theta)\sin\theta + (\varphi \cdot r \cdot \sin\theta)\cos\theta$$

$$= \varepsilon_x \cdot r \cdot \cos^2\theta + \varepsilon_y \cdot r \cdot \sin^2\theta \sin\theta + \varphi \cdot r \cdot \sin\theta \cdot \cos\theta$$

Since strain of the diagonal, $\varepsilon_\theta = C''C'/r$

$\therefore$

$$\varepsilon_\theta = \varepsilon_x \cdot \cos^2\theta + \varepsilon_y \cdot \sin^2\theta + \varphi \cdot \sin\theta \cdot \cos\theta \tag{2.28}$$

$$= \frac{1}{2}\varepsilon_x(1+\cos 2\theta) + \frac{1}{2}\varepsilon_y(1-\cos 2\theta) + \frac{1}{2}\varphi \sin 2\theta$$

$$= \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta \qquad (2.28a)$$

Compare the results with bi-axial and shear stresses conditions (Eq. 2.25).

- In a linear strain system, $\varepsilon_\theta = \varepsilon_x \cdot \cos^2\theta$ or $\varepsilon_x\left(\frac{1+\cos 2\theta}{2}\right)$
- In a pure shear system and for $\theta = 45°$, $\varepsilon_{45}{}^\circ = \varphi/2$.

Shear Strain

The shear strain at a point on a plane inclined at angle θ is the change in the angle between two straight lines perpendicular to each other. As shown in Fig. 2.37, if these lines are OC and OE before distortion, they become OC' and OE' after distortion. Let the angle between OC and OC' be α and between OE and OE' be γ.

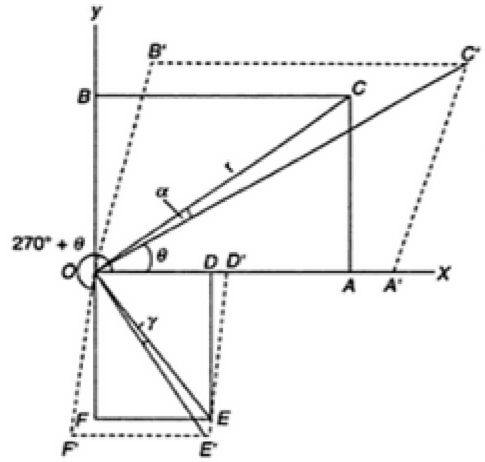


Fig. 2. 37

Thus, shear strain = φ_θ = change of angle of OC – change of angle of OE

$= \alpha - \gamma$

As the angle α is small, $\alpha \cong \tan\alpha = CC''/r$

$$CC' = CF + FC'' = CF + (GE - GH)$$
$$= (\varepsilon_x \cdot r \cdot \cos\theta)\sin\theta + [(\varphi \cdot r \cdot \sin\theta)\sin\theta - (\varepsilon_y \cdot r \cdot \sin\theta)\cos\theta]$$
$$= \varepsilon_x \cdot r \cdot \sin\theta \cdot \cos\theta + \varphi \cdot r \cdot \sin^2\theta - \varepsilon_y \cdot r \cdot \sin\theta \cdot \cos\theta$$
$$= \frac{1}{2}(\varepsilon_x - \varepsilon_y) r \cdot \sin 2\theta + \varphi \cdot r \cdot \sin^2\theta$$

$$\alpha = CC''/r = \frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \varphi \sin^2 6$$

Angle γ can be found by inserting $\theta = -(90° - \theta) = 270° + \theta$, in the above equation.

$$\gamma = \frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2(270° + \theta) + \varphi \sin^2(270° + \theta)$$
$$= \frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin(180° + 2\theta) + \varphi \sin^2(270° + \theta)$$
$$= -\frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \varphi \cos^2\theta$$

Shear strain $= \varphi_\theta = \alpha - \beta = (\varepsilon_x - \varepsilon_y)\sin 2\theta + \varphi(\sin^2\theta - \cos^2\theta)$

$= (\varepsilon_x - \varepsilon_y)\sin 2\theta - \varphi \cos 2\theta$

Compare the results with bi-axial and shear stress conditions (Eq. 2.26).

Lect-10

Principal strains and principal axes of strain measurements, Calculation of principal stresses from principal strains.

Principal strains

The maximum and the minimum values of strains on any plane at a point are known as the *principal strains* and the corresponding planes as the *principal planes for strains*.

To obtain the condition for principal strains, differentiating Eq. 2.40 with respect to θ and equating to zero,

$$\frac{d\varepsilon}{d\theta} = 0 - \frac{1}{2}(\varepsilon_x - \varepsilon_y)\, 2\sin 2\theta + \varphi \cdot \cos 2\theta$$

or

$$(\varepsilon_x - \varepsilon_y)\sin 2\theta = \varphi \cdot \cos 2\theta$$

or

$$\tan 2\theta = \frac{\varphi}{\sigma_x - \sigma_y} \tag{2.30}$$

Values of principal strains can be obtained in a similar way as for principal stresses:

$$\text{Principal strain } = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$

As $\tan 2\theta = \tan 2\theta = \dfrac{\varphi}{\sigma_x - \sigma_y}$, From Fig. 2.38,

$$\sin 2\theta = \pm \frac{\varphi}{\sqrt{(\varepsilon_y - \varepsilon_x)^2 + \varphi^2}}$$

$$\cos 2\theta = \pm \frac{\varepsilon_x - \varepsilon_y}{\sqrt{(\varepsilon_y - \varepsilon_x)^2 + \varphi^2}}$$

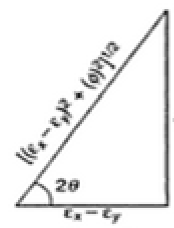


Fig. 2.38

Right-hand sides of both the above equations should have the same signs, positive or negative while using them.

- In principal planes, $\varphi_\theta = (\varepsilon_x - \varepsilon_y)\sin 2\theta - \varphi\cos 2\theta$

$$= (\varepsilon_x - \varepsilon_y)\frac{\varphi}{\sqrt{(\sigma_x - \sigma_y)^2 + \varphi^2}} - \varphi\frac{\varepsilon_x - \varepsilon_y}{\sqrt{(\sigma_x - \sigma_y)^2 + \varphi^2}} = 0$$

- It can be shown that the planes of principal strains are the same as of principal stresses as follows:

$$\tan 2\theta = \frac{\varphi}{\varepsilon_x - \varepsilon_y} = \frac{\tau/G}{(1/E)[(\sigma_x - v\sigma_y - v\sigma_z) - (\sigma_y - v\sigma_z - v\sigma_x)]}$$

$$= \frac{\tau \cdot E}{G[(\sigma_x - v\sigma_y - v\sigma_z) - (\sigma_y - v\sigma_z - v\sigma_x)]}$$

$$= \frac{\tau \cdot 2G(1+v)}{G(\sigma_x - \sigma_y)(1+v)} = \frac{2\tau}{\sigma_x - \sigma_y}$$

which is the same equation as Eq. 2.31 indicating that the planes of principal strains are the same as of principal stresses and thus can simply be referred as *principal planes*.

Lect-11
Mohr's circle for strain

Moh's strain circle

For the plane strain conditions can we derivate the following relations

$$\epsilon_\theta = \left\{\frac{\epsilon_x + \epsilon_y}{2}\right\} + \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \qquad (1)$$

$$\frac{1}{2}\gamma_\theta = -\left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right] \qquad (2)$$

Re writing the equation (1) as below :

$$\left[\epsilon_\theta - \left(\frac{\epsilon_x + \epsilon_y}{2}\right)\right] = \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \qquad (3)$$

squaring and adding equations (2) and (3)

$$\left[\epsilon_\theta - \left(\frac{\epsilon_x + \epsilon_y}{2}\right)\right]^2 + \left\{\frac{1}{2}\gamma_\theta\right\}^2 = \left[\left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta\right]^2 + \left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right]^2$$

$$\left[\epsilon_\theta - \left(\frac{\epsilon_x + \epsilon_y}{2}\right)\right]^2 + \left\{\frac{1}{2}\gamma_\theta\right\}^2 = \left(\frac{\epsilon_x + \epsilon_y}{2}\right)^2 + \frac{\gamma^2_{xy}}{4}$$

Now as we know that

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y$$

$$\left(\frac{\epsilon_1 - \epsilon_2}{2}\right)^2 = \left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \frac{\gamma^2_{xy}}{4}$$

Therefore the equation gets transformed to

$$\left[\epsilon_\theta - \left(\frac{\epsilon_1 + \epsilon_2}{2}\right)\right]^2 + \left[\frac{\gamma_\theta}{2}\right]^2 = \left(\frac{\epsilon_1 - \epsilon_2}{2}\right)^2 \qquad (4)$$

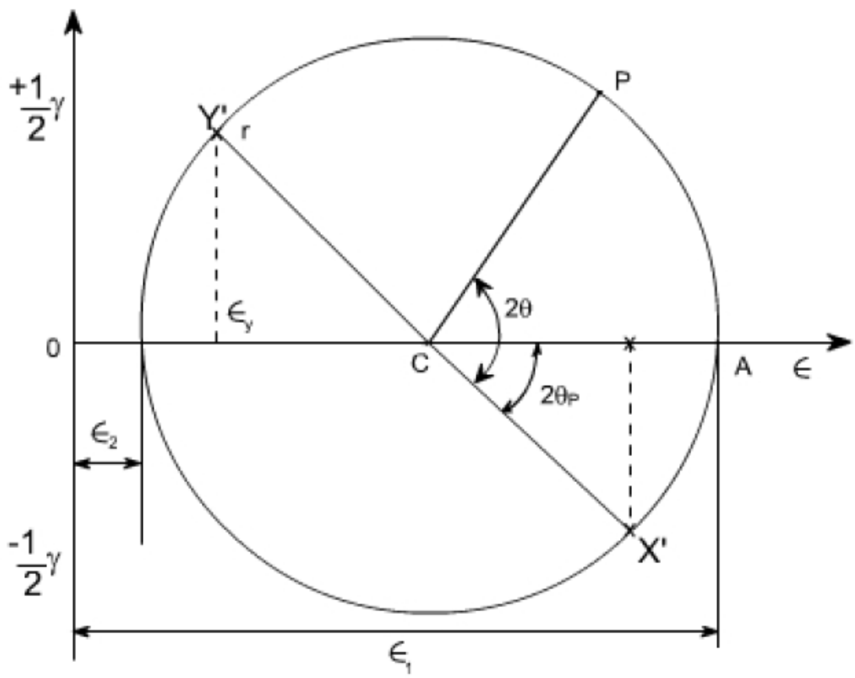
If we plot equation (4) we obtain a circle of radius $\left(\frac{\epsilon_1 - \epsilon_2}{2}\right)$ with center at $\left(\frac{\epsilon_1 + \epsilon_2}{2}, 0\right)$

A typical point P on the circle given the normal strain and half the sheer strain $1/2\gamma_{xy}$ associated with a particular plane. We note again that an angle subtended at the centre of Mohr's circle by an arc connecting two points on the circle is twice the physical angle in the material.

Mohr strain circle :

Since the transformation equations for plane strain are similar to those for plane stress, we can employ a similar form of pictorial representation. This is known as Mohr's strain circle.

The main difference between Mohr's stress circle and stress circle is that a factor of half is attached to the shear strains.



Points X' and Y' represents the strains associated with x and y directions with $\in$ and $\gamma_{xy}/2$ as co-ordiantes

Co-ordinates of X' and Y' points are located as follows :

$$X' = \left(\in_x, -\frac{\gamma_{xy}}{2}\right)$$

$$Y' = \left(\in_y, +\frac{\gamma_{xy}}{2}\right)$$

In x □ direction, the strains produced, the strains produced by σ_x,and $-\tau□_{xy}$ are $\in_x$ and $-\gamma_{xy}/2$

where as in the Y - direction, the strains are produced by $\in_y$ and + γ_{xy} are produced by σ_y and + τ_{xy}

These co-ordinated are consistent with our sign notation (i.e. + ve shear stresses produces produce +ve shear strain & vice versa)

on the face AB is τ_{xy}+ve i.e strains are ($\in_y$, +γ_{xy}/2) where as on the face BC,□τ_{xy} is negative hence the strains are (□$\in_x$, – γ_{xy} /2)

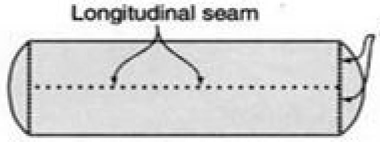
Lect-12
Stresses in thin cylinders under internal pressure

INTRODUCTION:

We devout our attention to **pressure vessels** in this chapter. Pressure vessels are containers that hold fluid under pressure. Examples are

- Tanks
- Boilers
- Pipelines
- Pressure cookers

We see tank trucks on the highways that contains all kinds of fluids, some of which are gases under relatively high pressures. We fill our automobile tyres at workshop. The gas for this comes from a storage tank of air maintained at high pressure. Automobile tyres themselves are pressure vessels. Failure of pressure vessels can cause loss of life either by sudden bursting or explosion or by simple failure such as leakage permitting lethal or highly explosive gases to escape into ambient atmosphere. Therefore, design of vessels containing fluids under pressure is to be carefully done to avoid mishaps. We concentrate on cylindrical-shaped pressure vessels only. Figure 8.1 shows a schematic view of a cylindrical pressure vessel.



THICK CYLINDER AND THIN CYLINDER:

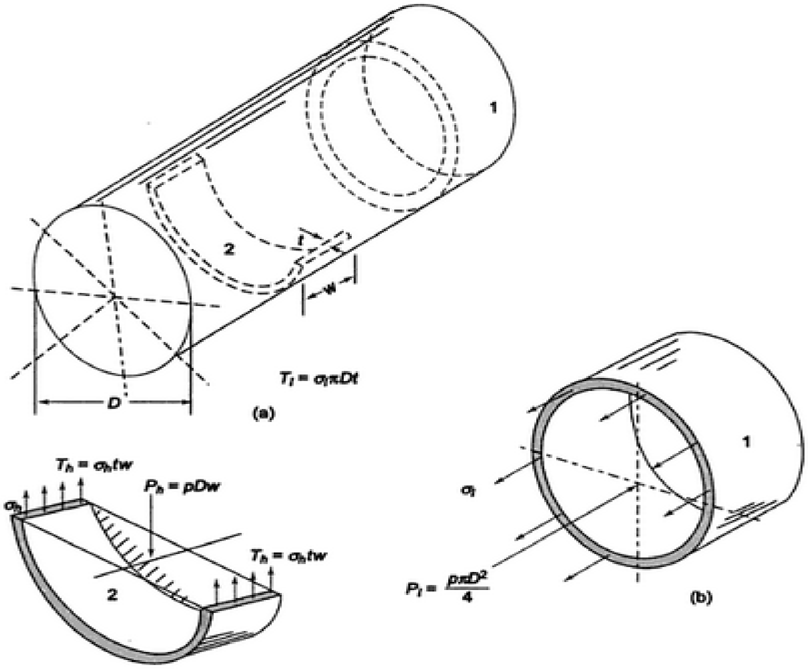
The walls of the pressure vessels are subjected to three-dimensional stress system (It may be recalled that in previous chapters, we studied only two-dimensional stress system). Up to a point the designer can assume that the stress is evenly distributed through the wall. The point up to which this assumption holds good is greatly influenced by the wall thickness of the vessel. Table 8.1 brings out certain differences between thin and thick cylinders.

Differences between thin and thick cylinders

Thin cylinder	Thick cylinder
The wall thickness is less than one-tenth of the inner radius of the cylinder.	The wall thickness is more than or equal to one-tenth of the inner radius of the cylinder.
The radial shear stress is neglected.	The radial shear stress is considered.
The hoop stress is assumed to be uniformly distributed over the thickness.	The hoop stress varies parabolically over the wall thickness.
Examples are tyres, gas storage tanks.	Examples are gun barrels, high-pressure vessels in oil-refining industry.
Analytical treatment for stresses is simple and approximate.	Analytical treatment is complex and accurate.
A thin cylinder is statically determinate.	A thick cylinder is statically indeterminate.
State of stress is membrane i.e. biaxial.	State of stress is triaxial.

Stress devlope in a thin cylinder:

Consider a thin walled cylinder as shown in Fig. 5.1(a). The cylinder is pressurized from inside and its ends are closed. Of course the simplifying assumption is that the distribution of stress on the wall thickness is uniform. A free body is cut by a plane normal to the axis and free body diagram of this portion is shown in Fig. 5.1(b). It is seen that on the free body an external force due to pressure acts along the axis of the cylinder. Call this force P_l where suffix pertains to length, indicating that P_l is a longitudinal force. The freebody is marked as 1.



The axial force P_l acts on the closed end. To bring the body 1 in equilibrium the stress will be induced on the thickness all along the circumference and force due to this stress which acts in the direction of the axis, will oppose the axial pressure force. This stress is called *longitudinal stress* and will be denoted by σ_l.

The diameter of cylinder = D
The thickness of cylinder = t
The pressure inside the cylinder = p
The longitudinal pressure force,

$$P_l = p\frac{\pi D^2}{4} \qquad \text{(i)}$$

I

The uniformly distributed stress on the thickness in the axial direction or longitudinal stress = σ_l
The longitudinal force due to σ_l across the cut circumferential area,

$$T_l = \sigma_l \pi D t \qquad \text{(ii)}$$

P_l and T_l acting along the length keep the free body in equilibrium. Hence

$$T_l = P_l$$

Using (i) and (ii)

$$\sigma_l \pi D t = p\frac{\pi D^2}{4}$$

$$\therefore \quad \sigma_l = \frac{pD}{4t} \qquad (5.1)$$

Apparently σ_l is tensile

A second free body marked 2 is cut from the cylinder of Fig. 5.1(a) by a plane which contains the axis. This free body is shown in Fig. 5.1(c). The width of the free body is w and the fluid will exert a force P_h as shown in Fig. 5.1(c). To counteract force P_h tensile stress σ_h will be induced uniformly on the thickness as shown in the Fig. 5.1(c). σ_h will cause a force T_h on each length segment of the cylinder.

$$P_h = pwD \quad \text{(iii)}$$

$$T_h = \sigma_h tw \quad \text{(iv)}$$

This is due to the fact that pressure acting on a curved surface causes a force which is equal to the product of pressure and projected area of the curved surface.

For equilibrium of free body 2, $P_h = 2T_h$

$$\therefore \quad 2\sigma_h tw = pwD$$

or

$$\sigma_h = \frac{pD}{2t} \quad (5.2)$$

It is seen that the stress σ_h acts along the circumference. It is known as *circumferential stress* or *hoop stress*.

It is also seen from Eqns. (5.1) and (5.2) that

$$\sigma_h = 2\sigma_l$$

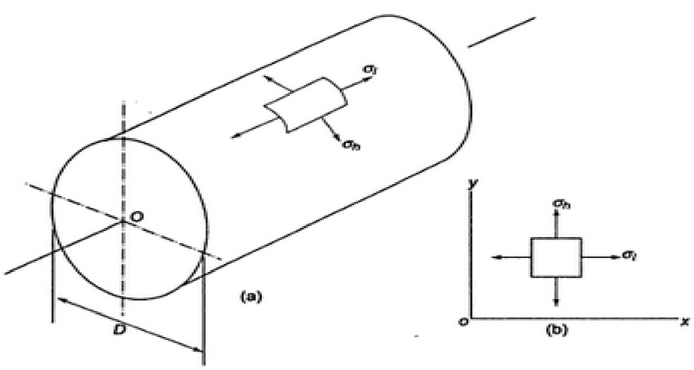
So the failure will occur due to σ_h i.e., along the length. For this reason the longitudinal joint is required to be made stronger than the circumferential joint.

Thus a thin cylinder pressurized from inside is subjected to two stresses, respectively along length and circumference. Both of them are tensile in nature. These stresses will cause the volume to increase.

Strain and change in volume

The state of stress at any point in a thin cylinder is completely described by two stress components σ_l and σ_h. The state of stress is referred to x and y axes where x is parallel to the cylinder axis and y is perpendicular to this axis and passes through the point of interest.

The state of stress is shown in Fig. 5.2.



From Eqn. (3.4), the strain along the length

$$\varepsilon_l = \frac{1}{E}(\sigma_l - v\sigma_h) \quad \text{(i)}$$

Also strain along circumference

$$\varepsilon_h = \frac{1}{E}(\sigma_h - v\sigma_l) \quad \text{(ii)}$$

where E is modulus of elasticity and v is Poisson's ratio for the material of the cylinder.

Using Eqns. (5.1) and (5.2)

$$\varepsilon_l = \frac{1}{E}\left(\frac{pD}{4t} - \nu\frac{pD}{2t}\right) \tag{5.3}$$

$$= \frac{pD}{4tE}(1 - 2\nu)$$

$$\varepsilon_h = \frac{1}{E}\left(\frac{pD}{2t} - \nu\frac{pD}{4t}\right)$$

$$= \frac{pD}{4tE}(2 - \nu) \tag{5.4}$$

Let C_1 and C be the circumference of the cylinder respectively after application of pressure and before application of pressure. Let D_1 and D be the corresponding diameters. Then

$$\varepsilon_h = \frac{\text{Change in circumference}}{\text{Original circumference}}$$

$$= \frac{\pi D_1 - \pi D}{\pi D}$$

$$= \frac{D_1 - D}{D}$$

or

$$\varepsilon_h = \frac{\text{Change in diameter}}{\text{Original diameter}}$$

In other words, the circumferential or hoop strain in a pressurized thin cylinder is equal to diametral strain.

i.e.

$$\frac{dD}{D} = \varepsilon_h \tag{5.5}$$

Now consider the volume of the cylinder, V

$$V = \frac{\pi}{4}D^2 l$$

Taking total differential of both sides

$$dV = \frac{\pi}{4}.2D\,(dD)l + \frac{\pi}{4}D^2\,(dl)$$

Dividing left hand side by V and right hand side by $\frac{\pi}{4}D^2 l$

$$\frac{dV}{V} = 2\frac{dD}{D} + \frac{dl}{l}$$

From basic definition [as given by Eqn. (5.5)] $\frac{dD}{D}$ is diametral strain and $\frac{dl}{l}$ is linear strain in the cylinder.

Hence

$$\frac{dV}{V} = 2\,\varepsilon_h + \varepsilon_l \tag{5.6}$$

It must be noted here that V is the volume of the space in the cylinder and not the volume of the material which makes the cylinder. V will be the volume of any fluid contained in the cylinder. Thus dV will be the change in volume of the fluid which is filled in the cylinder or $\frac{dV}{V}$ will be the *volumetric strain* in the fluid that pressurizes the cylinder.

Using Eqns. (5.3) and (5.4) in Eqn. (5.6)

$$\frac{dV}{V} = \frac{2pD}{4tE}(2-v) + \frac{pD}{4tE}(1-2v)$$

$$= \frac{pD}{4tE}(4-2v+1-2v)$$

$$= \frac{pD}{4tE}(5-4v)$$

or change in the volume of fluid filled in a cylinder of diameter D under a pressure of p,

$$dV = V\frac{pD}{4tE}(5-4v)$$

Using Eqns. (5.3) and (5.4) in Eqn. (5.6)

$$\frac{dV}{V} = \frac{2pD}{4tE}(2-v) + \frac{pD}{4tE}(1-2v)$$

$$= \frac{pD}{4tE}(4-2v+1-2v)$$

$$= \frac{pD}{4tE}(5-4v)$$

or change in the volume of fluid filled in a cylinder of diameter D under a pressure of p,

$$dV = V\frac{pD}{4tE}(5-4v) \tag{5.7}$$

Lect-13
thin spherical shells under internal pressure, stress in cylindrical shell with hemispherical ends

THIN SPHERICAL SHELL:

Spherical storage tanks are also used frequently. The thin shell will satisfy the same condition for distribution of stress on the thickness as the thin cylinder, that is stress over thickness is uniform. For a sphere of internal diameter D and thickness t the ratio D/t must be greater than 20 to satisfy the condition of uniform distribution of stress across the thickness and then the sphere is treated as thin. Apparently because of symmetry of spherical shell in all directions the stress at any point will have equal components and both will be tensile if pressure acts from inside. A sphere and state of stress at a point are shown in Fig. 5.12(a).

A hemispherical free body in which the thickness along a circle of diameter equal to the diameter of sphere is shown in Fig. 5.12(b). The internal force developed on the thickness will act perpendicularly as shown. The bursting force P is derived from pressure.

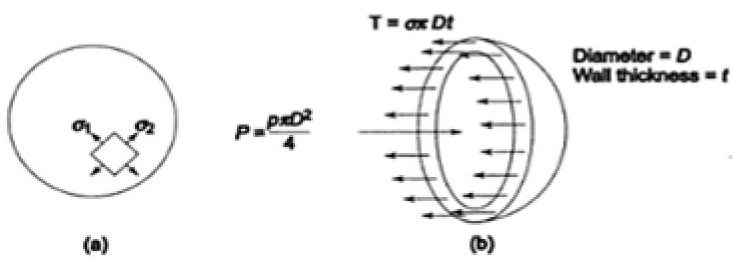


Fig. 5.12

Let p be the internal pressure
D be the diameter of sphere and
t its thickness.

Then $$P = \frac{p\pi D^2}{4} \quad \text{(i)}$$

$$\therefore \quad \sigma_1 = \frac{pD}{4t} \quad (5.13)$$

If another section perpendicular to the one shown in Fig. 5.12(b) is taken and another hemispherical free body is considered the stress on the exposed circular periphery will be perpendicular to σ_1. Call this stress σ_2. Equating the bursting force with the force caused by σ_2 will result in

$$\sigma_2 = \frac{pD}{4t} \quad (5.14)$$

Thus $$\sigma_1 = \sigma_2 = \frac{pD}{4t}$$

Hence at any point in the shell of the sphere two mutually perpendicular direct stresses will be acting. There is no shearing stress associated with σ_1 and σ_2 as the conditions of equilibrium are satisfied. Hence these two are principal stresses. Since both σ_1 and σ_2 act along the spherical surface, they are also called hoop stresses and may be denoted by σ_h. As σ_1 is acting away from the exposed circular section of the hemisphere in Fig. 5.11(b) it is tensile in nature.

Finding maximum shearing stress in this situation may be tricky. One must remember that at any point on the thickness in a third direction which is radial must be considered for stress acting in this direction will contribute to maximum shearing stress, τ_{max}

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 0 \quad (5.15)$$

will give maximum shearing stress in the plane of the shell only. However, remembering that the outer surface (Fig. 5.12) is free from any stress then in a radical plane at any point 2 on the outside of the shell there exist two principal stresses σ_1 and zero giving

$$\tau_{max\,2} = \frac{\sigma_1 - 0}{2} = \frac{\sigma_1}{2} \tag{5.16}$$

Also remembering that on the inner surface pressure is acting and thus in a radial plane at a point 3 on the inside of a shell there exist two principal stresses σ_1 and $-p$; giving

$$\tau_{max\,3} = \frac{\sigma_1 - (-p)}{2} = \frac{\sigma_1 + p}{2}$$

$$= \frac{pD}{2 \times 4t} + \frac{p}{2}$$

$$= \frac{pD}{4t} \times \frac{1}{2}\left[1 + \frac{4t}{D}\right]$$

$$= \frac{\sigma_1}{2}\left[1 + \frac{4t}{D}\right] \tag{5.17}$$

$\tau_{max\,3}$ will be equal to $\tau_{max\,2}$ if $\frac{4t}{D}$ is negligible in comparison to 1. This is possible for very thin shells in which $t/D < \frac{1}{20}$. In the limiting case of thin shell where t/D is just 1/20, $4t/D = 0.2$ and if it is neglected the maximum shearing stress will be underestimated by 16.7%. Similar, analysis can be done for thin cylinders also.

Figure 5.13 illustrates the positions of points 2 and 3 on the outside and inside surfaces of a spherical shell element *A* through which intersects a radial plane *S*. This radial plane produces a section *abcd* in the element. The line *bc* is on the outside and the line *ad* is inside the element. Point 2 is on *bc* whereas point 3 is on *ad*. The states of stress at points 2 and 3 in the plane *S* are also indicated in Fig. 5.13.

Volume Change in Spherical Shell

The volume of a sphere of diameter D

$$V = \frac{1}{6}\pi D^3$$

or

$$V = \frac{4}{3}\pi R^3$$

If R is the radius.

$\therefore$

$$\frac{dV}{dD} = \frac{\pi}{2} D^2$$

or

$$\frac{dV}{V} = \frac{\pi}{2} \frac{D^2 \times dD}{\frac{\pi}{6} D^3}$$

i.e.

$$\frac{dV}{V} = 3\frac{dD}{D} = \frac{3dR}{R} \tag{5.18}$$

which means that volumetric strain in a thin spherical pressurized vessel is three times the diametral or radial strain.

It is not difficult to show that the hoop strain in the sphere is equal to radial or diametral strain. It was shown in case of a thin cylinder in Sec. 5.2. Thus

$$\varepsilon_1 = \varepsilon_h = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$$

or

$$\varepsilon_h = \frac{\sigma_1}{E}(1-\nu) = \frac{dD}{D}$$

$\therefore$

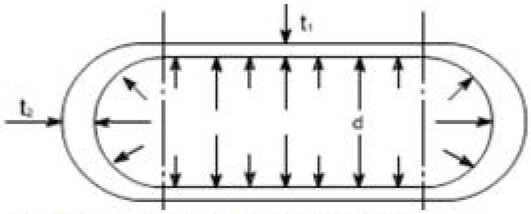
$$\frac{dV}{V} = \frac{3\sigma_1}{E}(1-\nu)$$

$$= \frac{3pD}{4tE}(1-\nu)$$

Cylindrical Vessel with Hemispherical Ends:

Let us now consider the vessel with hemispherical ends. The wall thickness of the cylindrical and hemispherical portion is different. While the internal diameter of both the portions is assumed to be equal

Let the cylindrical vassal is subjected to an internal pressure p.



For the Cylindrical Portion

hoop or circumferential stress $= \sigma_{HC}$ 'c' here synifies the cylindrical portion.

$$= \frac{pd}{2t_1}$$

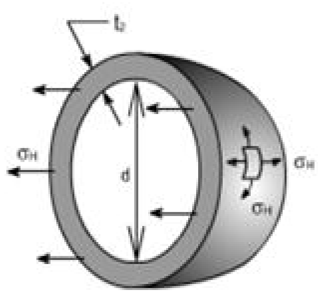
longitudnal stress $= \sigma_{LC}$

$$= \frac{pd}{4t_1}$$

hoop or circumferential strain $\in_2 = \frac{\sigma_{HC}}{E} - \nu\frac{\sigma_{LC}}{E} = \frac{pd}{4t_1E}[2-\nu]$

or $\boxed{\in_2 = \frac{pd}{4t_1E}[2-\nu]}$

For The Hemispherical Ends:



Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoops or circumferential stresses of equal values. Again the radial stresses are neglected in comparison to the hoop stresses as with this cylinder having thickness to diametre less than1:20.

Consider the equilibrium of the half □ sphere

Force on half-sphere owing to internal pressure = pressure x projected Area

$= p. \pi d^2/4$

$$\text{Resisting force} = \sigma_H . \pi.d.t_2$$

$$\therefore \quad p.\frac{\pi.d^2}{4} = \sigma_H . \pi d.t_2$$

$$\Rightarrow \sigma_H \text{ (for sphere)} = \frac{pd}{4t_2}$$

$$\text{similarly the hoop strain} = \frac{1}{E}\left[\sigma_H - \nu.\sigma_H\right] = \frac{\sigma_H}{E}[1-\nu] = \frac{pd}{4t_2E}[1-\nu] \text{ or } \boxed{\epsilon_{2s} = \frac{pd}{4t_2E}[1-\nu]}$$

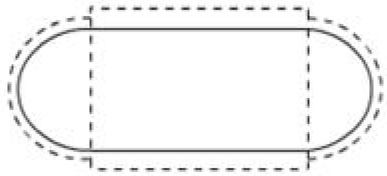


Fig □ shown the (by way of dotted lines) the tendency, for the cylindrical portion and the spherical ends to expand by a different amount under the action of internal pressure. So owing to difference in stress, the two portions (i.e. cylindrical and spherical ends) expand by a different amount. This incompatibly of deformations causes a local bending and sheering stresses in the neighborhood of the joint. Since there must be physical continuity between the ends and the cylindrical portion, for this reason, properly curved ends must be used for pressure vessels.

Thus equating the two strains in order that there shall be no distortion of the junction

$$\frac{pd}{4t_1E}[2-\nu] = \frac{pd}{4t_2E}[1-\nu] \text{ or } \frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

But for general steel works $\nu = 0.3$, therefore, the thickness ratios becomes

$t_2 / t_1 = 0.7/1.7$ or

$$\boxed{t_1 = 2.4\ t_2}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispheroid ends for no distortion of the junction to occur.

Lect-14
wire winding of thin cylinders

wire winding of thin cylinders

Whenever a thin pipe is wound with a wire under tension, at a close pitch, compressive stresses will be initially set up in the pipe section. If, now, the pipe is subjected to internal fluid pressure, the bursting force will be resisted by the pipe as well as the wire each offering tensile stresses. The final hoop stress in the pipe material will be the hoop stress due to internal fluid pressure minus the initial compressive stress. Hence the final stress will be less than if the pipe was unwound one.

The final tensile stress in the wire is equal to the tensile stress due to the internal fluid pressure plus the initial tensile winding stress since the wire is initially under tension. Thus the wire wound pipe will be stronger as it will carry a greater internal pressure for a given permissible tensile stress. Consider a cylinder of diameter D and wall thickness t around which a wire of diameter d is wound closely under a tensile stress of σ_w. The cylinder is subjected to an internal fluid pressure p.

Let E_w = Modulus of elasticity for the material of wire

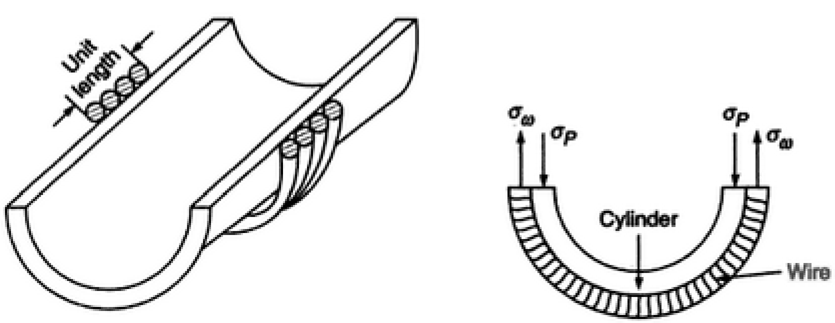
E_p = Modulus of elasticity for the material of pipe

Let us consider a unit length of the cylinder. Number of turns of the wire per unit length of cylinder $= \dfrac{1}{d}$.

Cross-sectional area of the winding wire (both sides) per unit length of cylinder $= 2 \times \dfrac{\pi}{4} d^2 \times \dfrac{1}{d} = \dfrac{\pi d}{2}$.

If σ_w is the tensile stress in the winding wire without fluid pressure inside the cylinder, then total tensile force in the wire $\dfrac{\pi d}{2} \times \sigma_w$.

This tensile force induces a compressive stress σ_p in the cylinder.



(a) (b)

Total cross-sectional area subjected to compression = $2 \times t \times 1 = 2t$.

Total compressive force in cylinder resisting the total tension of the wire = $2t \times \sigma_p$

$$\therefore \quad 2t \times \sigma_p = \frac{\pi d}{2} \times \sigma_w$$

or

$$\sigma_p = \frac{\pi d}{4t} \times \sigma_w$$

When the fluid is admitted inside the cylinder the bursting force is resisted by newly developed tensile stresses σ_p^1 and σ_w^1 in the cylinder and in the wire respectively.

Bursting force = Resisting force of pipe + resisting force of wire rings.

$$p \times D \times 1 = \sigma_p^1 \times 2t \times 1 + \sigma_w^1 \times \frac{\pi d}{2}$$

$$pD = \sigma_p^1 \times 2t + \sigma_w^1 \times \frac{\pi d}{2} \quad \text{(i)}$$

Initial strain of cylinder (without internal pressure) due to winding of wire

$$= \frac{\sigma_p}{E_p} \quad \text{(ii)}$$

If σ_L is the longitudinal stress then final strain in the cylinder

$$= \frac{\sigma_p^1}{E_p} - \frac{1}{m} \times \frac{\sigma_L}{E_p} \quad \text{(iii)}$$

$\therefore$ Change in strain on the surface of cylinder

$$= \frac{\sigma_p^1}{E_p} - \frac{1}{m} \times \frac{\sigma_L}{E_p} - \frac{\sigma_p}{E_p}$$

$$= \frac{1}{E_p}\left[\sigma_p^1 - \frac{\sigma_L}{m} - \sigma_p\right] \quad \text{(iv)}$$

But change in strain of winding wire due to internal pressure

$$= \frac{\sigma_w^1}{E_w} - \frac{\sigma_w}{E_w} = \frac{1}{E_w}\ (\sigma_w^1 - \sigma_w)$$

Since the changes in strains in the cylinder and in the winding wire are equal

$$\frac{1}{E_p}\left[\sigma_p^1 - \frac{\sigma_L}{m} - \sigma_p\right] = \frac{1}{E_w}\ (\sigma_w^1 - \sigma_w) \quad \text{(v)}$$

If σ_{w_2} and σ_{p_2} be the final stress in the wire and in the cylinder, then we have

$$\sigma_{w_2} = \sigma_w + \sigma_w^1 \text{ both are tensile}$$

and

$$\sigma_{p_2} = \sigma_p^1 - \sigma_p\ \ \sigma_p^1 \text{ is tensile and } \sigma_p \text{ is compressive.}$$

Lect-15
Shear force and bending moment. Types of load and Types of support. Support reactions, Relationship between bending moment and shear force

INTRODUCTION OF BEAM AND TYPES OF BEAM :

Beam is a structural component which carries the loads transverse to its longitudinal axis and is supported at its two ends. The longitudinal axis of a beam is the line that joins the centroidal points of all the transverse sections along its length. In beams the loads, the reactions and the longitudinal axis of the beam, all lie in one plane, called the plane of bending. Sometimes the loads on the beam may not be truly transverse to its axis and so in that case, in addition to the transverse loads, the beam will have axial force also. The section of the beam along its axis may be prismatic or non-prismatic. Prismatic beams have transverse sections that remain uniform throughout their spans. Non-prismatic beams, on the other hand, have sections that may vary in accordance with certain geometric pattern along their span lengths or that may vary in steps or may vary in an at-random fashion. The distance between the centre to centre of the supports is called the span of the beam and the clear distance between the supports is defined as its clear span.

Depending on the way the beams are supported, they are categorised as:

(1) *Cantilever Beam:* This is a beam which has one end fixed in position and direction and the other end is free. The fixed support of the cantilever is also called an encaster or a built-in support. At the

fixed end of a cantilever beam three reactions can develop corresponding to the three displacement components that this support checks and these reactions are necessary and sufficient to support any one or more loads acting on the cantilever and maintain its stability. At the free end no reaction develops as at that end no restraint is provided either to the rotation or to the translation that may occur there.

(2) *Simple supported beam:* A beam, having one end hinged and the other end supported on a roller or a knife edge, is called a simply supported beam. As a roller support can provide restraint against one vertical movement only one vertical reacstion is possible at this type of the support. On the other hand, since, a hinge restrains both the horizontal and the vertical movements at the support point it gives rise to two reactions correspondingly, i.e., one horizontal and the other vertical. So in a simply supported beam a total of three reactions, i.e., two vertical and one horizontal, develop at the two supports. These three reactions are necessary and sufficient to support all the possible load combinations likely to act on the beam.

(3) *Overhanging beam:* A simply supported beam, with one of its ends overhanging on one side of a support or both of its ends overhanging beyond the corresponding supports, is called an overhanging beam. For the same overall length of a beam, the overhangs reduce the effective span of the beam and so make the design of the beam economical when compared to the design of a simply supported beam of the same length under the same loads.

One thing that is common to the type of the beams, discussed above, is that in all of them there is a single span, two supports and three independent reaction components that depend on the type and the magnitude of the loads acting on the beams. Also all the loads and the reactions acting on the beams form a coplanar system of forces. Since this system needs three equations of statics to be satisfied for its equilibrium the three independent reaction components acting on the beam, as a free body, can be directly obtained from the three equations of equilibrium. Such beams are, therefore, statically determinate. These beams are shown in Fig. 4.1.1.

Beams in which the number of the supports or their nature is such that they impart more than three reactions then such beams become externally indeterminate. Some such beams are:

(a) *Propped cantilever:* It is a cantilever beam which is propped or simply supported at the free end. This type of the beam, therefore, has one fixed support and the other roller or a knife edge support. Since in all four reactions develop at the two supports of the beam and only three equations of equilibrium are available for analysing such beams, by considering them as a free body under a general loading system, a propped cantilever becomes external indeterminate beam of the first order.

(b) *Fixed, encastre or built-in beam:* This beam has both of its ends fixed in position as well as in direction. In all six reactions develop in this beam at both of its supports and so its external indeterminacy becomes two under parallel vertical load systems and three under any other general load systems.

(c) *Continuous beam:* A beam, having more than one span or more than three supports, is a continuous beam. More the number of intermediate supports, more and more will be the reactions. For stability and equilibrium of a continuous beam, a minimum of one of the supports must be hinged and all others supports must be the simple supports or the roller supports. A two span continuous beam, as such, has four reactions and that makes it externally indeterminate beam of the first degree.

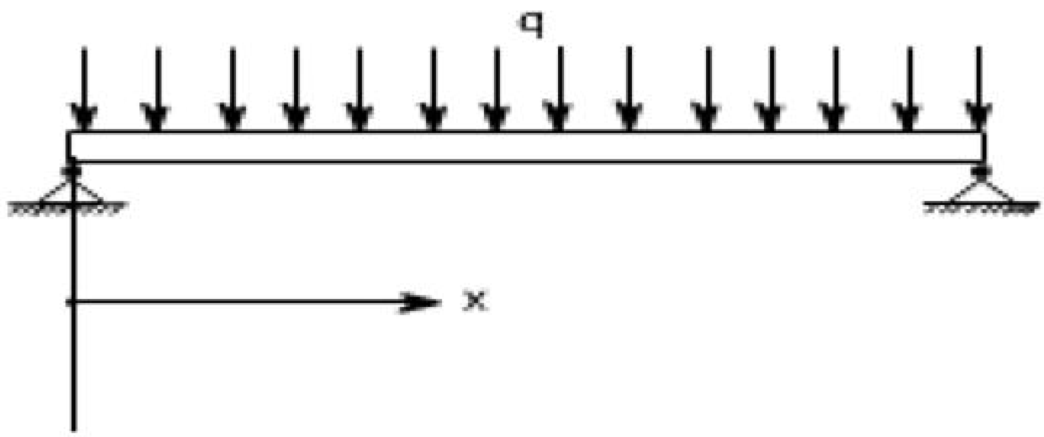
TYPES OF LOADS :

Types of loads acting on beams:

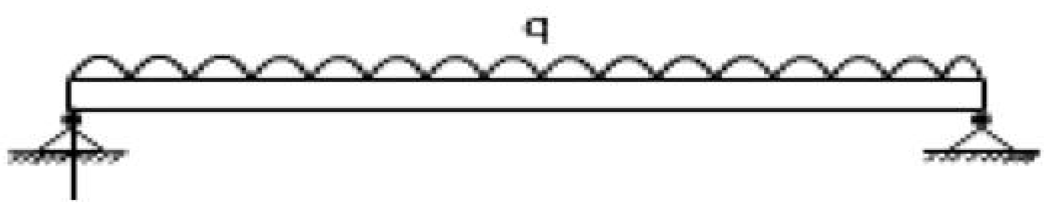
A beam is normally horizontal where as the external loads acting on the beams is generally in the vertical directions. In order to study the behaviors of beams under flexural loads. It becomes pertinent that one must be familiar with the various types of loads acting on the beams as well as their physical manifestations.

A. Concentrated Load: It is a kind of load which is considered to act at a point. By this we mean that the length of beam over which the force acts is so small in comparison to its total length that one can model the force as though applied at a point in two dimensional view of beam. Here in this case, force or load may be made to act on a beam by a hanger or though other means

B. Distributed Load: The distributed load is a kind of load which is made to spread over a entire span of beam or over a particular portion of the beam in some specific manner

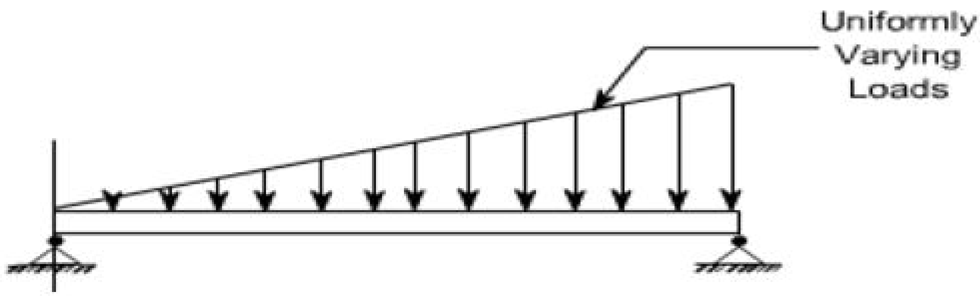


OR



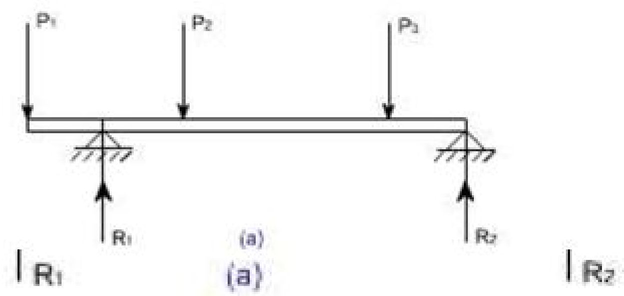
UNIFORM VARYING LOAD

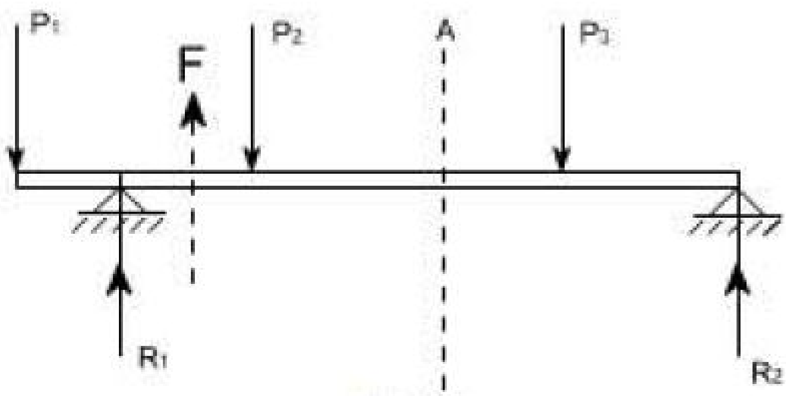
some times the load acting on the beams may be the uniformly varying as in the case of dams or on inclind wall of a vessel containing liquid, then this may be represented on the beam as below:



Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms



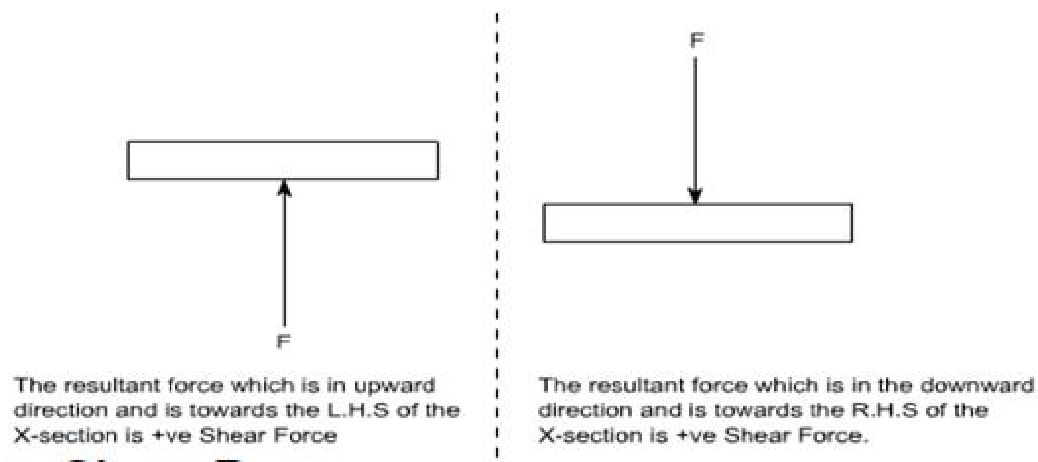


Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is tc divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is □F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces □F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

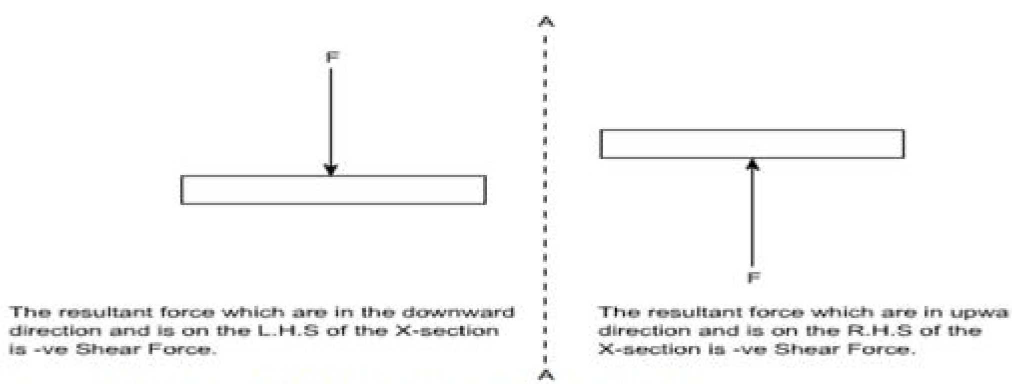
Therefore, now we are in a position to define the shear force □F' to as follows:

At any x-section of a beam, the shear force □F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

SIGN CONVECTION

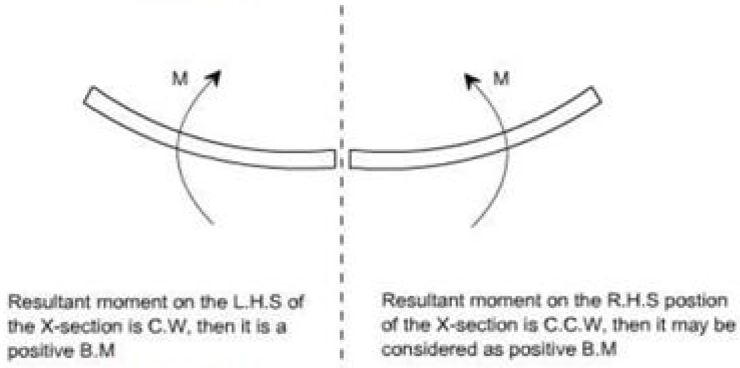


Positive Shear Force

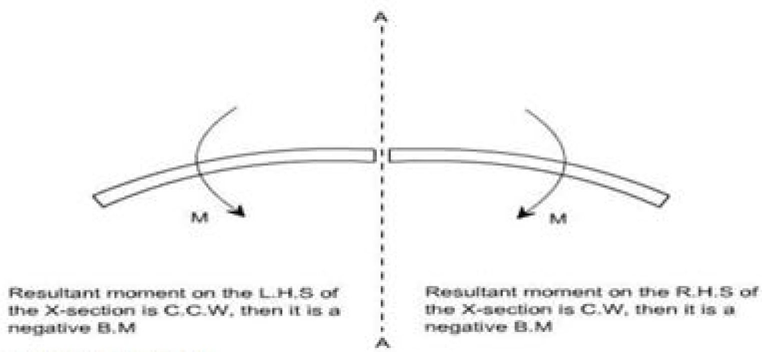


Negative Shear Force

Sign convection of Bending moment



Positive Bending Moment



Negative Bending Moment

Some times, the terms □Sagging' and Hogging are generally used for the positive and negative bending moments respectively.

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

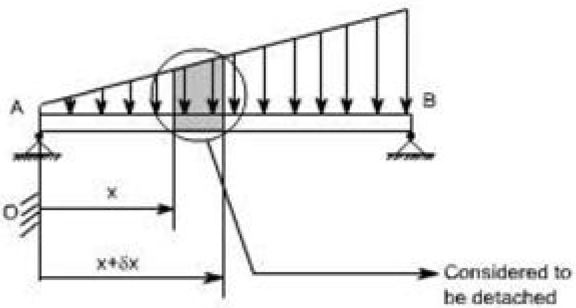
Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force □F' varies along the length of beam. If x dentotes the length of the beam, then F is function x i.e. F(x).

Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment □M' varies along the length of the beam. Again M is a function x i.e. M(x).

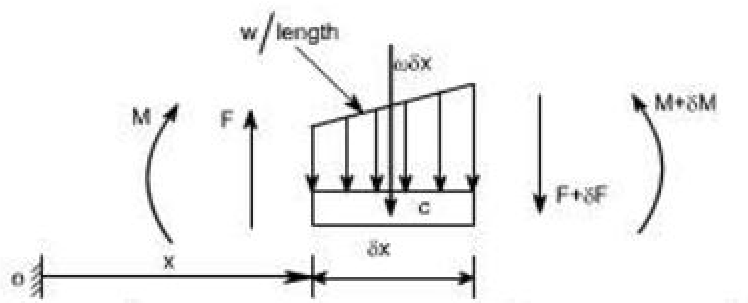
, **Relationship between bending moment and shear force**

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance x' from the origin 0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ δF at the section x and x + δx respectively.
- The bending moment at the sections x and x + δx be M and M + dM respectively.
- Force due to external loading, if w' is the mean rate of loading per unit length then the total loading on this slice of length δx is w. δx, which is approximately acting through the centre c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre c'.

This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M$$ [Neglecting the product of δF and δx being small quantities]

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots (2)$$

Lect-16
Shear Force and Bending Moment diagrams of cantilever beam carrying point loads,UDL,uvl and related problem

1. **Draw the shearforce and bending moment diagram of cantilever beam subjected to point load W at free end**

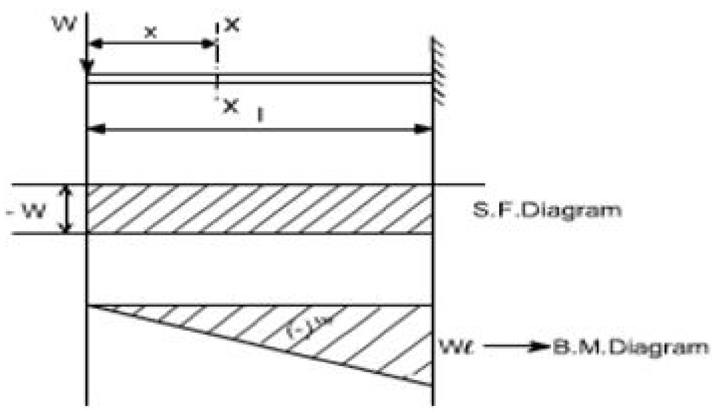
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative

Taking moments about the section gives (obviously to the left of the section)

M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise directi and is therefore taken as □ve according to the sign convention)

so that the maximum bending moment occurs at the fixed end i.e. M = -W l

From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



2. **Draw the shearforce and bending moment diagram of cantilever beam subjected to udl on whole span**

Let the load be distributed over the whole length of the beam, the loading being w per unit run (Fig. 9).

Consider the section XX at a distance x from the free end A.

S.F. at $X = S_x = -wx$

B.M. at $X = M_x = -w \cdot x \cdot \frac{x}{2} = -\frac{wx^2}{2}$

Thus we find that the variation of the shear-force is according to a *linear law*, while the variation of bending moment is according to *parabolic law*.

At $x = 0$, $S_x = 0$ and $M_x = 0$

At $x = l$, $S_x = -wl$ and $M_x = -\frac{wl^2}{2}$.

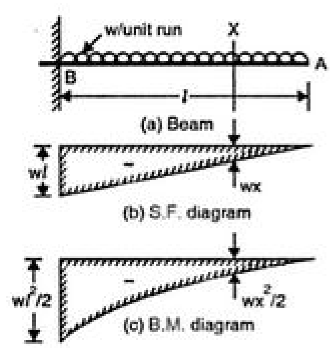


Fig. 9

3.SF and BM of cantilever beam subjected to uniform varing load

Fig. 14 shows a cantilever AB of length l carrying a load whose intensity varies uniformly from zero at the free end to w per unit run at the fixed end. Let the intensity of loading at XX, at a distance x from the free end A be w_x per unit run.

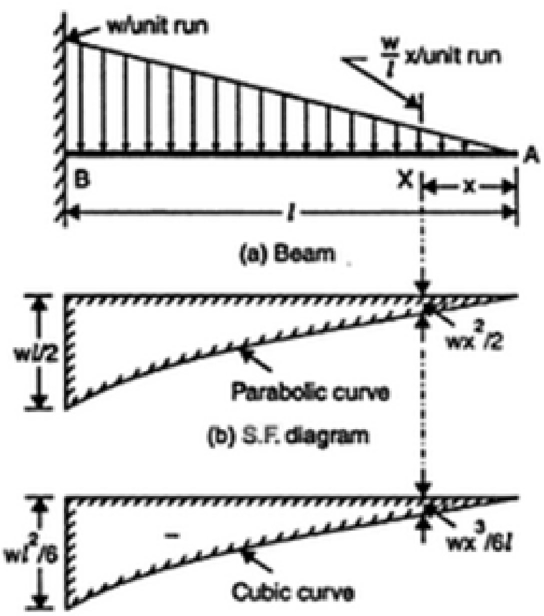
$\therefore \quad w_x = \frac{w}{l} . x$ since the intensity of load increases uniformly from zero at the free end to w at the fixed.

$\therefore$ Load acting for an elemental distance dx from $x = w_x . dx$, thus the total load acting for any distance between $x = a$ and $x = b$.

$$= \sum_{x=a}^{x=b} w_x . dx$$

= area of load diagram between $x = a$ and $x = b$.

Hence we arrive at an important conclusion that the *total distributed load acting on any segment equals the area of the load diagram on that segment.*



S.F. and B.M. at distance x from the end A are given by

S_x = Area of the load diagram between X and A

$$= -\frac{1}{2} . x . w_x = -\frac{1}{2} . x . \frac{w}{l} . x = -\frac{wx^2}{2l}$$

M_x = Moment of the load acting on XA about X

= Area of load diagram between X and A × distance of centroid of this diagram from X

$$= -\frac{wx^2}{2l} . \frac{x}{3} = -\frac{wx^3}{6l}$$

At $x = 0$, $S_x = 0$ and $M_x = 0$

At $x = l$, $S_x = -\frac{wl}{2}$ and $M_x = -\frac{wl^2}{6}$

The S.F. changes following a *parabolic law* while the B.M. changes following a *cubic law.*

Lect-17
Shear Force and Bending Moment diagrams of simple supported beam beam carrying point loads,udl

Shear Force and Bending Moment diagrams of simple supported beam beam carrying point loads

2.5.1. The Point Load is at the Mid-point of Simply Supported Beam. Fig. 2.25 shows a beam AB of length L simply supported at the ends A and B and carrying a point load W at its middle point C.

The reactions at the support will be equal to $\frac{W}{2}$ as the load is acting at the middle point of the beam. Hence $R_A = R_B = \frac{W}{2}$.

Take a section X at a distance x from the end A between A and C.

The reactions at the support will be equal to $\frac{W}{2}$ as the load is acting at the middle point of the beam. Hence $R_A = R_B = \frac{W}{2}$.

Take a section X at a distance x from the end A between A and C.

Let F_x = Shear force at X,

and M_x = Bending moment at X.

Here we have considered the *left portion* of the section. The shear force at X will be equal to the resultant force acting on the left portion of the section. But the resultant force on the left portion is $\frac{W}{2}$ acting upwards. But according to the sign convention, the resultant force on the *left portion* acting upwards is considered positive. Hence shear force at X is positive and its magnitude is $\frac{W}{2}$.

$$\therefore \qquad F_x = +\frac{W}{2}$$

Hence the shear force between A and C is constant and equal to $+\frac{W}{2}$.

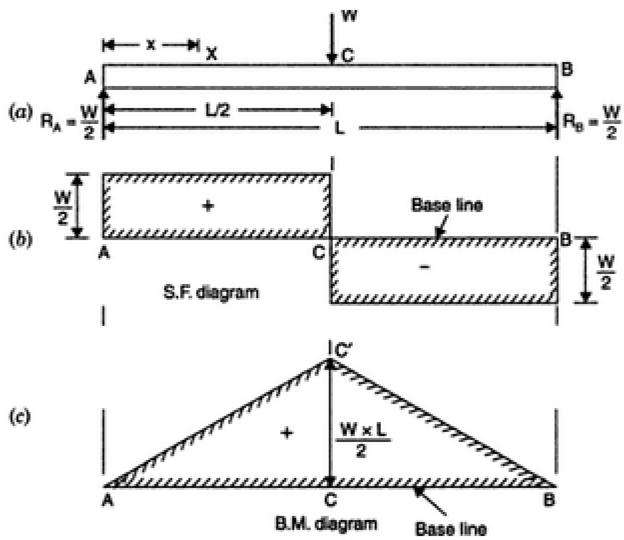
Now consider any section between C and B at distance x from end A. The resultant force on the left portion will be

$$\left(\frac{W}{2} - W\right) = -\frac{W}{2}.$$

This force will also remain constant between C and B. Hence shear force between C and B is equal to $-\frac{W}{2}$.

At the section C the shear force changes from $+\frac{W}{2}$ to $-\frac{W}{2}$.

The shear force diagram is shown in Fig



Bending Moment Diagram

(*i*) The bending moment at any section between *A* and *C* at a distance of *x* from the end *A*, is given by

$$M_x = R_A.x \quad \text{or} \quad M_x = +\frac{W}{2}.x \qquad \text{...(i)}$$

(B.M. will be positive as for the *left portion* of the section, the moment of all forces at *X* is clockwise. Moreover, the bending of beam takes place in such a manner that concavity is at the top of the beam).

At *A*, $x = 0$ hence $M_A = \frac{W}{2} \times 0 = 0$

At *C*, $x = \frac{L}{2}$ hence $M_C = \frac{W}{2} \times \frac{L}{2} = \frac{W \times L}{4}$.

From equation (*i*), it is clear that B.M. varies according to straight line law between *A* and *C*. B.M. is zero at *A* and it increases to $\frac{W \times L}{4}$ at *C*.

(*ii*) The bending moment at any section between *C* and *B* at a distance *x* from the end *A*, is given by

$$M_x = R_A.x - W \times \left(x - \frac{L}{2}\right)$$

$$= \frac{W}{2}.x - Wx + W \times \frac{L}{2} = \frac{WL}{2} - \frac{2x}{2}$$

At *C*, $x = \frac{L}{2}$ hence $M_C = \frac{WL}{2} - \frac{W}{2} \times \frac{L}{2} = \frac{W \times L}{4}$

At *B*, $x = L$ hence $M_B = \frac{WL}{2} - \frac{W}{2} \times L = 0.$

Hence bending moment at *C* is $\frac{WL}{4}$ and it decreases to zero at *B*. Now the B.M. diagram can be completed as shown in Fig. 2.25 (*c*).

SF AND BM OF SIMPLE SUPPORTED BEAM CARRYING POINT LOAD AT SOME DISTANCE FROM SOME END BEAM

a beam AB of length L, simply supported at the ends A and B and carrying a point load W at C at a distance of 'a' from the end A.

Let R_A = Reaction at the support A, and

R_B = Reaction at the support B.

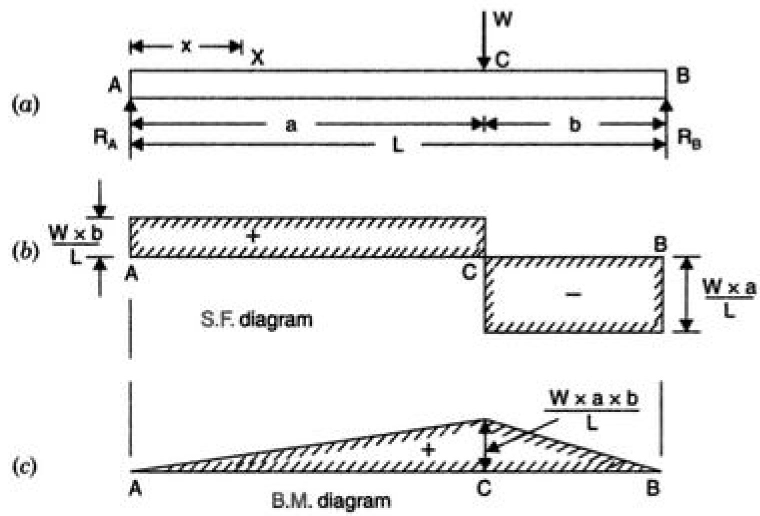
First calculate the reactions, by taking moments about A or about B.

Taking moments of the forces on the beam about A, we get

$$R_B \times L = W \times a$$

$$\therefore \quad R_B = \frac{W.a}{L} \quad \text{and} \quad R_A = W - R_B = W - \frac{W.a}{L}$$

$$= W\left(1 - \frac{a}{L}\right) = W\left(\frac{L-a}{L}\right) = \frac{W \times b}{L} \qquad (\because \; L - a = b)$$



Consider a section X at a distance x from the end A between A and C.

The shear force F_x at the section is given by,

$$F_x = +R_A = +\frac{W.b}{L} \qquad ...(i)$$

(The shear force will be positive as the resultant force on the *left portion* of the section is acting upwards).

The shear force between A and C is constant and equal to $\frac{W.b}{L}$.

Now consider any section between C and B at a distance x from the end A. The resultant force on the left portion will be $R_A - W$

or

$$\frac{W.b}{L} - W = W.\left(\frac{b-L}{L}\right) = -W\left(\frac{L-b}{L}\right) = -\frac{W.a}{L} \qquad (\because \; L - b = a)$$

The shear force between C and B is constant and equal to $-\frac{W.a}{L}$. At the section C, the shear force changes from $\frac{W.b}{L}$ to $-\frac{W.a}{L}$. The shear force diagram is shown in Fig. 2.26 (b).

Bending Moment Diagram

(*i*) The bending moment at any section between A and C at a distance x from the end A, is given by

$$M_x = R_A \times x = +\frac{W.b}{L}.x \qquad \text{(Plus sign due to sagging)}$$

At A, $x = 0$ hence $M_A = \frac{W.b}{L} \times 0 = 0$

At C, $x = a$ hence $M_C = \frac{W.b}{L}.a = \frac{W.a.b}{L}$

Hence the B.M. increases from zero at A to $\frac{W.a.b}{L}$ at C by a straight line law. The B.M. is zero at B. Hence B.M. will decrease from $\frac{W.a.b}{L}$ at C to zero at B following a straight line law. The B.M. diagram is drawn in Fig. 2.26 (*c*).

From the shear force and bending moment diagrams, it is clear that the B.M. is maximum at C where the S.F. changes its sign.

SF AND BM OF SIMPLE SUPPORTED BEAM CARRYING UDL IN WHOLE SPAN

A BEAM AB OF LENGH L SIMPLY supported at the ends A and B and carrying a uniformly distributed load of w per unit lengtl over the entire length. The reactions at the supports will be equal and their magnitude will b half the total load on the entire length.

Let R_A = Reaction at A, and

R_B = Reaction at B

$\therefore \quad R_A = R_B = \frac{w.L}{2}$

Consider any section X at a distance x from the *left end* A. The shear force at the sectior (*i.e.*, F_x) is given by,

$$F_x = +R_A - w.x = +\frac{w.L}{2} - w.x \qquad ...(i$$

From equation (*i*), it is clear that the shear force varies according to straight line law. The values of shear force at different points are :

At A, $x = 0$ hence $\quad F_A = +\frac{w.L}{2} - \frac{w.0}{2} = +\frac{w.L}{2}$

At B, $x = L$ hence $\quad F_B = +\frac{w.L}{2} - w.L = -\frac{w.L}{2}$

At C, $x = \frac{L}{2}$ hence $\quad F_C = +\frac{w.L}{2} - w.\frac{L}{2} = 0$

The shear force diagram is drawn as shown in Fig. 2.28 (*b*).

The bending moment at the section X at a distance x from left end A is given by,

$$M_x = +R_A.x - w.x.\frac{x}{2}$$

$$= \frac{w.L}{2}.x - \frac{w.x^2}{2} \qquad \left(\because R_A = \frac{w.L}{2}\right) ...(ii)$$

From equation (*ii*), it is clear that B.M. varies according to parabolic law.

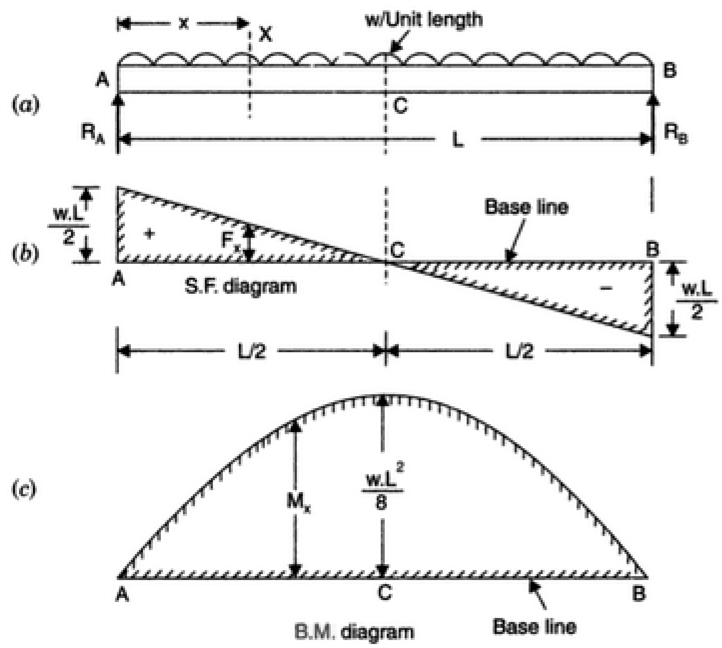
The values of B.M. at different points are :

At A, $x = 0$ hence $M_A = \frac{w.L}{2} . 0 - \frac{w.0}{2} = 0$

At B, $x = L$ hence $M_B = \frac{w.L}{2} . L - \frac{w}{2} . L^2 = 0$

At C, $x = \frac{L}{2}$ hence $M_C = \frac{w.L}{2} \cdot \frac{L}{2} - \frac{w}{2} \cdot \left(\frac{L}{2}\right)^2$

$$= \frac{w.L^2}{4} - \frac{w.L^2}{8} = + \frac{w.L^2}{8}$$



Thus the B.M. increases according to parabolic law from zero at A to $+\frac{w.L^2}{8}$ at the middle point of the beam and from this value the B.M. decreases to zero at B according to the parabolic law.

Lect-18

Shear Force and Bending Moment diagrams simple supported UvL,moment

Shear Force and B.M. Diagrams for a Simply Supported Beam Carrying a Uniformly Varying Load from Zero at One End to w Per Unit Length at the Other End. Fig. 2.34 shows a beam AB of length L simply supported at the ends A and B and carrying a uniformly varying load from zero at end A to w per unit length at B. First calculate the reactions R_A and R_B.

Taking moments about A, we get

$$R_B \times L = \left(\frac{w \cdot L}{2}\right) \cdot \frac{2}{3} L \qquad \left[\text{Total load} \left(= \frac{w \cdot L}{2}\right) \text{is acting } \frac{2}{3} L \text{ from } A\right]$$

Fig. 18.35

$\therefore$

$$R_B = \frac{w \cdot L}{3}$$

and

$$R_A = \text{Total load on beam} - R_B = \frac{w \cdot L}{2} - \frac{w \cdot L}{3} = \frac{w \cdot L}{6}$$

Consider any section X at a distance x from end A. The shear force at X is given by,

$$F_x = R_A - \text{load on length } AX = \frac{w \cdot L}{6} - \frac{w \cdot x}{L} \cdot \frac{x}{2}$$

$$\left(\text{Load on } AX = \frac{AX \cdot CX}{2} = \frac{x \cdot w \cdot x}{2 \cdot L}\right)$$

$$= \frac{wL}{6} - \frac{wx^2}{2L} \qquad \text{...}(i)$$

Equation (i) shows that S.F. varies according to parabolic law.

At A, $x = 0$ hence, $\quad F_A = \frac{w \cdot L}{6} - \frac{w}{2L} \times 0 = \frac{w \cdot L}{6}$

At B, $x = L$, hence, $\quad F_B = \frac{w \cdot L}{6} - \frac{w \cdot L^2}{2L} = \frac{w \cdot L}{6} - \frac{w \cdot L}{2} = \frac{w \cdot L - 3w \cdot L}{6} = -\frac{2w \cdot L}{6} = -\frac{w \cdot L}{3}$

The shear force is $+\frac{wL}{6}$ at A and it decreases to $-\frac{wL}{3}$ at B according to parabolic law. Somewhere between A and B, the S.F. must be zero. Let the S.F. be zero to a distance x from A. Equating the S.F. to zero in equation (i), we get

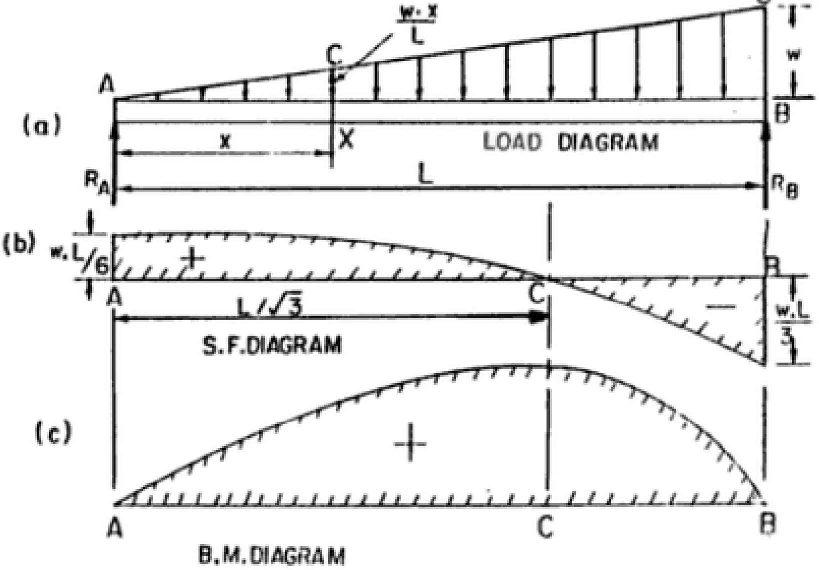
$$0 = \frac{wL}{6} - \frac{wx^2}{2L} \quad \text{or} \quad \frac{wx^2}{2L} = \frac{wL}{6}$$

or

$$x^2 = \frac{wL}{6} \times \frac{2L}{w} = \frac{L^2}{3}$$

$\therefore$

$$x = \frac{L}{\sqrt{3}} = 0.577\, L$$



B.M. Diagram

The B.M. is zero at A and B.

The B.M. at the section X at a distance x from the end A is given by,

$$M_x = R_A.x - \text{Load on length } AX \cdot \frac{x}{3} \qquad \left(\because \text{ Load on } AX \text{ is acting at } \frac{x}{3} \text{ from } X\right)$$

$$= \frac{w.L}{6} \cdot x - \frac{wx^2}{2L} \cdot \frac{x}{3} = \frac{wL}{6} \cdot x - \frac{wx^3}{6L}$$

Equation (*ii*) shows the B.M. varies between A and B according to cubic law.

Max. B.M. occurs at a point where S.F. becomes zero after changing its sign.

That point is at a distance of $\frac{L}{\sqrt{3}}$ from A. Hence substituting $x = \frac{L}{\sqrt{3}}$ in equation (*ii*), we get maximum B.M.

$$\therefore \qquad \text{Max. B.M.} = \frac{w.L}{6} \cdot \frac{L}{\sqrt{3}} - \frac{w}{6L} \cdot \left(\frac{L}{\sqrt{3}}\right)^3$$

$$= \frac{wL^2}{6\sqrt{3}} - \frac{wL^2}{18\sqrt{3}} = \frac{3w \cdot L^2 - wL^2}{18\sqrt{3}} = \frac{wL^2}{9\sqrt{3}}.$$

Simply Supported Beam Subjected to External Moment M_0 at $x = a$ from Left Support

Consider the beam AB of span L subjected to an external clockwise moment at a point, distance 'a' from support A as shown in Fig. 3.13a.

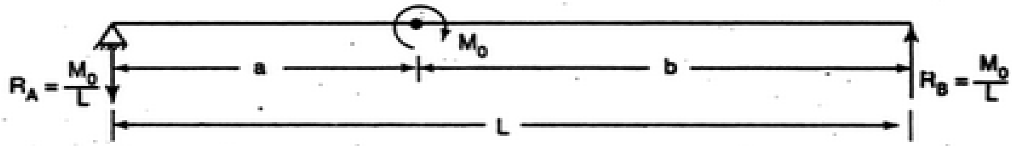
Taking moment about B, and downward reaction at A as positive

$$R_A L - M_0 = 0 \quad \text{or} \quad R_A = \frac{M_0}{L} \quad \text{(Note: } R_A \text{ is downward)}$$

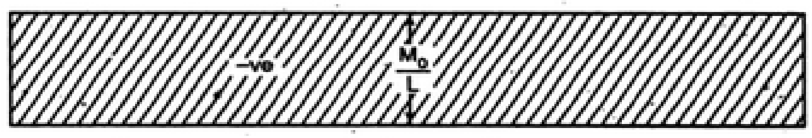
$$\therefore \quad R_B = R_A = \frac{M_0}{L} \quad \text{(Upward)}$$

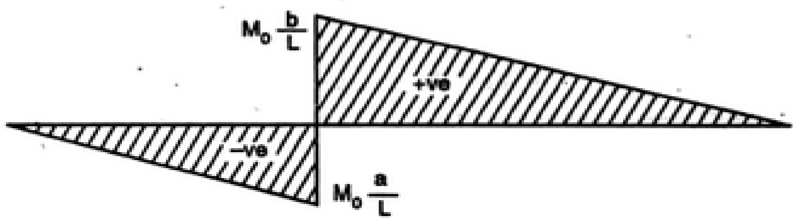
Now at any section at distance x from A

$$F = -R_A = -\frac{M_0}{L} \quad \text{(Constant)}$$



(a) Space Diagram





(c) BMD

Hence *SFD* is shown in Fig. 3.13b.

At section x—x, in portion left of the section.

$$M = -R_A x = -\frac{M_0}{L}x \quad \text{(Linear variation)}$$

At $x = 0$, $\quad M = 0$

At $x = a$, $\quad M = -\frac{M_0 a}{L}$

To the right of the section x—x,

$$M = -\frac{M_0}{L}x + M_0 = M_0\left[1 - \frac{x}{L}\right] \quad \text{(Linear variation)}$$

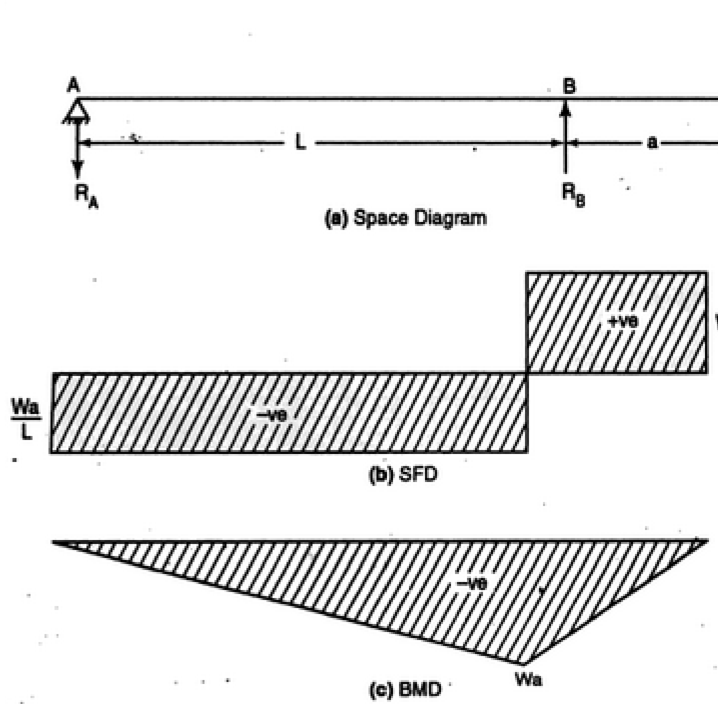
$\therefore$ At $x = a$, $\quad M = M_0\left[1 - \frac{a}{L}\right] = \frac{M_0 b}{L}$

and at $x = L$, $\quad M = M_0\left[1 - \frac{L}{L}\right] = 0$

Lect-19
Shear Force and Bending Moment diagrams of over hang beam carrying point loads, UDL, moment and point of inflection

Overhanging Beam Subjected to Concentrated Load at Free End

Let *ABC* be the overhanging beam subjected to a concentrated load *W* at free end as



(a) Space Diagram

(b) SFD

(c) BMD

Taking moment equilibrium condition about *B*, we get

$$R_A L = Wa \qquad \text{[Note: } R_A \text{ is downward]}$$

Portion AB: Measuring *x* from *A* and considering left hand side forces, we get

$$F = -R_A = -\frac{Wa}{L} \qquad \text{(Constant)}$$

For portion *AB*, *SFD* is negative

$$M = -R_A x$$

$$= -\frac{Wa}{L}x \qquad \text{(Linear variation)}$$

At $x = 0$, $\quad M = 0$

At $x = L$, $\quad M = -\frac{Wa}{L}L = -Wa$

BMD for this portion can be drawn.

Portion BC: Measuring *x* from free and *C*,

$$F = W \qquad \text{(Constant)}$$

∴ *SFD* can be drawn for this portion.

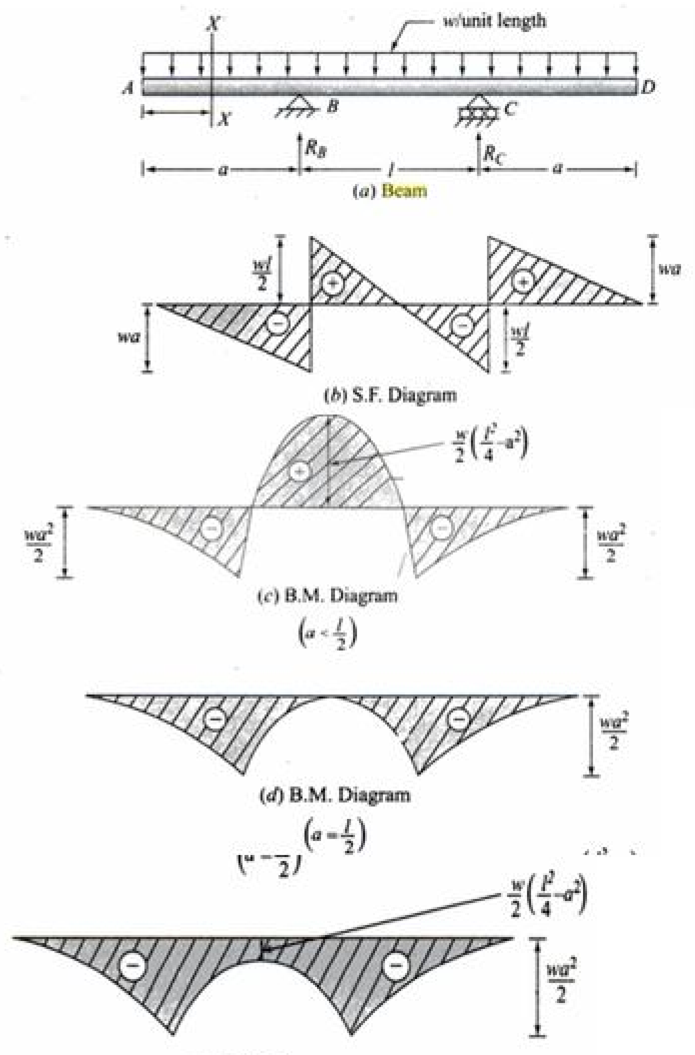
$$M = -Wx \qquad \text{(Linear variation)}$$

At $x = 0$, $\quad M = 0$

At $x = a$, $\quad M = -Wa$

BMD for this portion also can be drawn.

Overhanging beam with equal overhangs on each side and loaded with a uniformly distributed load over its entire span



(e) B.M. Diagram

Reactions at *B* and *C*

Take moments of forces about *B*.

$$w \times a \times \frac{a}{2} + R_C \times l = w\,(l + a)\,\frac{(l+a)}{2}$$

$$\frac{wa^2}{2} + R_C l = \frac{w}{2}\,[l^2 + a^2 + 2la]$$

or $$R_C = \frac{w\,(l+2a)}{2}\,(\uparrow)$$

But $R_B + R_C = w\,(l + 2a)$ or $R_B = \frac{w\,(l+2a)}{2}\,(\uparrow)$

Calculations for shear force

Consider a section *XX* of the beam at a distance *x* from *A*. Shear force at the section is

$$F_x = -wx$$

Shear force at *A*,

$$F_A = 0 \quad (\text{for } x = 0)$$

Shear force at B,

$$F_B = -wa \quad (\text{for } x = a)$$

Shear force just to the right of *B*

$$= -wa + R_B$$

$$= -wa + \frac{w}{2}(l + 2a)$$

$$= \frac{w}{2}l$$

Shear force at *C* is

$$F_C = \frac{w}{2}l - wl$$

$$= -\frac{wl}{2}$$

Shear force just to the right of *C*

$$= -\frac{wl}{2} + R_C$$

$$= -\frac{wl}{2} + \frac{w}{2}(l + 2a)$$

$$= wa$$

Shear force at *D* is

$$F_D = wa - wa = 0$$

Calculations for bending moment

Bending moment at the section is

$$M_x = -w.x.\frac{x}{2}$$

Bending moment at *A*, $M_A = 0 \quad (\text{for } x = 0)$

Bending moment at B is

$$M_B = -w \cdot a \cdot \frac{a}{2}$$

$$= -\frac{wa^2}{2}$$

2

Bending moment at C is

$$M_C = -w(a+l)\frac{(a+l)}{2} + R_B \times l$$

$$= -\frac{w}{2}(a+l)^2 + \frac{w}{2}(l+2a) \times l$$

$$= -\frac{w}{2}a^2$$

Bending moment at D is

$$M_D = 0$$

Shear force is zero at a distance $\left(a + \frac{l}{2}\right)$ from A. It is the position of maximum bending moment.

The maximum bending moment is given as

$$M_{max} = -w\left(a+\frac{l}{2}\right)\cdot\left(a+\frac{l}{2}\right)\cdot\frac{1}{2} + R_B\left(a+\frac{l}{2}-a\right)$$

$$= -\frac{w}{2}\left(a+\frac{l}{2}\right)^2 + \frac{w}{2}(l+2a)\cdot\frac{l}{2}$$

$$= \frac{w}{2}\left(\frac{l^2}{4} - a^2\right)$$

Case I

When $a < \frac{l}{2}$, then M_{max} will be positive.

Case II

When $a = \frac{l}{2}$, then $M_{max} = 0$

$$\frac{w}{2}\left(\frac{l^2}{4} - a^2\right) = 0$$

Lect-20

Theory of simple bending of initially straight beams, Bending stresses ,section modulus

BENDING Theory

It is interesting to note that **Galileo** [1564–1642] was one of the first scientists to make a theoretical investigation of the problem of bending. He attempted to determine the breaking load for a cantilever in terms of the cross sectional dimensions of the beam. **Charles Augustin de Coulomb** [1736–1806], a French engineer, was first to explain the correct relationship between the bending moment and the moment of resistance of a beam. The formulation of simple bending theory (also termed **engineer's** bending **theory**) is due to French scientists **Euler** and **Bernoulli**. The theory is based on some simplifying assumptions but has been found to yield satisfactory results.

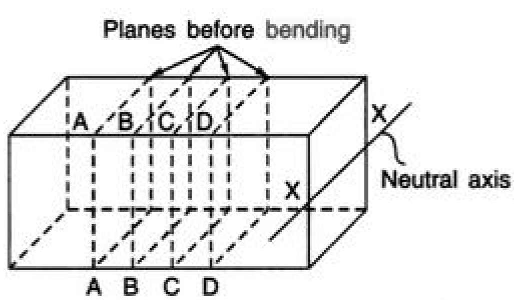
ASSUMPTION OF BENDING THEORY

The assumptions made in simple bending theory are

1. The material of the beam is homogeneous, isotropic and has the same properties in tension and compression.
2. The material obeys Hooke's law.
3. The beam is initially straight and bends into an arc of a circle.
4. The radius of curvature is large when compared to the span of the beam.
5. The beam bends with respect to one of the principal axis.
6. The beam is subjected to pure moment and deformations owing to shear are neglected.
7. Transverse sections of the beam that are plane before bending remain plane and normal to the beam axis even after bending.

All the assumptions are self explanatory; however, the last assumption needs little clarification. This assumption emphasizes that the planes such as AA, BB, CC and DD (Figure 4.9) before bending remains to be planes even after bending. That is, each plane revolves about its intersection with neutral plane xx. The meaning of this assumption is hidden in the fact that the deformations

All the assumptions are self explanatory; however, the last assumption needs little clarification. This assumption emphasizes that the planes such as AA, BB, CC and DD (Figure 4.9) before bending remains to be planes even after bending. That is, each plane revolves about its intersection with neutral plane xx. The meaning of this assumption is hidden in the fact that the deformations

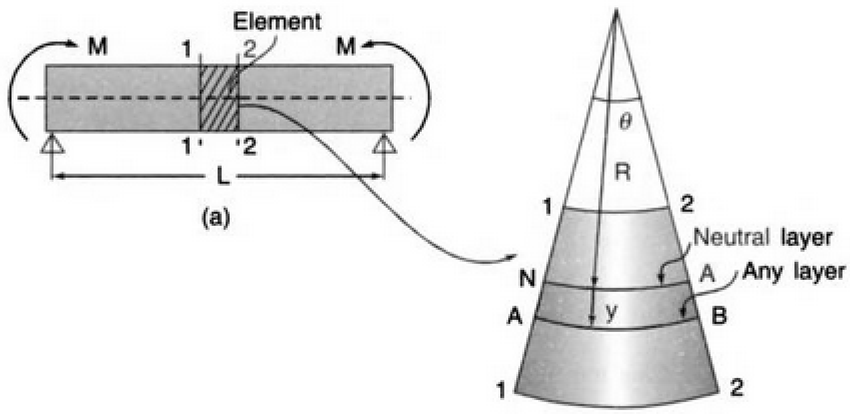


and consequently the stresses must increase uniformly from zero at the neutral plane to a maximum at the outer fibres. If this assumption is not realized, what we get is warped planes as shown in

FIGURE

DERIVATION OF BENDING EQUATION

Consider a beam of span L, subjected to concentrated moments at the ends as shown in Figure 4.11(a) (we are aware that this type of loading will enable the beam to be in **pure bending**). Consider a portion of the beam bounded between two sections 1-1 and 2-2. This segment is blown-up in Figure 4.11(b) for clarity. Consider any layer AB located at a height of y from neutral axis.



(b) Blown-up element

The strain in this layer owing to bending of the beam is

ε = (Change in length of the layer AB)/(Original length of the layer AB)

$$\varepsilon = \frac{(\text{AB} - \text{NA})}{\text{NA}} = \frac{(R + y)\theta - R\theta}{R\theta}$$

$$\varepsilon = \frac{y}{R}$$

Since the material obeys Hooke's law, stress should be proportional to strain

$$f \propto \varepsilon$$

$$f = E\varepsilon$$

∴

$$\varepsilon = \frac{f}{E}$$

$$\frac{f}{y} = \frac{E}{R}$$

or

$$\frac{f}{y} = \frac{E}{R} \qquad (4.1)$$

Further, we consider the fibre AB to possess an elementary area *da*, on this area bending stress f would prevail (refer Figure 4.12).

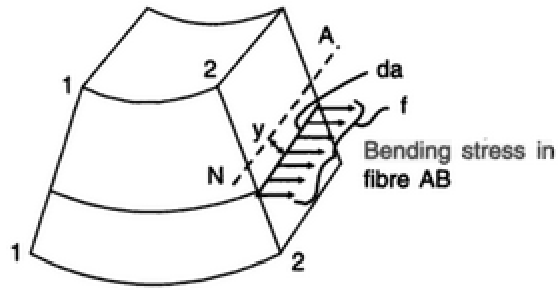


Figure 4.12 A Portion of Bent beam.

Force exerted on this fibre = $f\,da$.
Moment of this force about neutral axis = $f\,da \cdot y$

$$= \frac{E}{R} da \cdot y^2 \quad \left\{\because \quad f = \frac{E}{R} \cdot y\right\}$$

Moment generated by the entire area of cross section = $\frac{E}{R}\int y^2 da = \frac{E}{R} I$

The above expression is nothing but the moment of resistance developed by the section. For the equilibrium of the section, the moment of resistance thus developed must be equal to bending moment acting at the section due to external loads and reaction. Therefore

$$M = \frac{E}{R} I$$

or,

$$\frac{M}{I} = \frac{E}{R} \tag{4.2}$$

Combining Eqs. (4.1) and (4.2): $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$

The above equation is the famous bending equation (**Euler-Bernoulli's** equation). The usual notations are

M = Bending Moment at a section (N-mm).
I = Moment of Inertia of the cross section of the beam about Neutral axis (mm^4).

f = Bending stress in a fibre located at distance y from neutral axis (N/mm^2). This stress could be f_{bc} (bending compressive stress) or f_{bt} (bending tensile stress) depending on the location of the fibre.
y = Distance of the fibre under consideration from neutral axis (mm).
E = Young's Modulus of the material of the beam (N/mm^2).
R = Radius of curvature of the bent beam (mm).

The normal stresses in the beam are related to bending moment by considering part of the bending equation.

$$f = \frac{My}{I} \tag{4.3}$$

The above equation is called the **flexure formula**. This formula infers that *the bending stress in a fibre is directly proportional to its distance from neutral axis.*

FLEXURAL PARAMETER

1. ***Section Modulus:*** The maximum tensile and compressive stresses in the beam occur at points located farthest from the neutral axis. Let us denote y_1 and y_2 as the distances from the neutral axis to the extreme fibres at the top and the bottom of the beam. Then the maximum bending normal stresses are

$$f_{bc} = \frac{My_1}{I} = \frac{M}{I/y_1} = \frac{M}{Z_t},$$ where f_{bc} is bending compressive stress in the topmost layer.

Similarly

$$f_{bt} = \frac{My_2}{I} = \frac{M}{I/y_2} = \frac{M}{Z_b};$$ where f_{bt} is bending tensile stress in the bottom most layer.

Here, Z_t and Z_b are called **section moduli** of the cross sectional area, and they have dimensions of length to the third power (ex. mm^3). If the cross section is symmetrical (like rectangular or square sections), then $Z_t = Z_b = Z$, and Z is called as section modulus. *Section modulus is defined as the ratio of rectangular moment of inertia of the section to the distance of the remote layer from the neutral axis.* Thus, general expression for bending stress reduces to

$$f = \frac{M}{Z}$$

It is seen from the above expression that for a given bending moment, it is in the best interests of the designer of the beam to procure high value for section modulus so as to minimise the bending stress. More the section modulus designer provides for the beam, less will be the bending stress generated for a given value of bending moment.

2. ***Flexural Rigidity:*** If we consider the other part of the bending equation, i.e.

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad M = \frac{EI}{R}$$

The product ***EI*** is called the *flexural rigidity* of the beam. The moment sustained by an element of a beam is proportional to *EI*. The larger the value of *EI*, the larger will be the moment sustained by the beam. It is an index of flexural strength of the beam.

MODULUS OF RUPTURE

The flexure formula, Eq. (4.3) may be used to determine the bending stress in a beam loaded to failure (or rupture) in a testing machine. Since the proportional limit of the material will be exceeded while breaking, the stress predicted by the formula is not true stress. Nevertheless, the imaginary stress so obtained is called modulus of rupture. It may be defined as the *maximum bending stress required to cause the bending failure of a beam.* It is used as a yardstick to compare ultimate bending strength of beams of various sizes and materials.

Lect-21
Position of neutral axis ,Bending stress developed in different section

Position of neutral axis

Consider a cross section as shown in Figure 4.13 of a simply supported beam. Let the area of this cross section be A. Let *da* represent the elementary area of a layer located at a height of *y* from the neutral axis. Let *M* be the bending moment acting on the section considered. The consequent bending stress on this area be *f*.

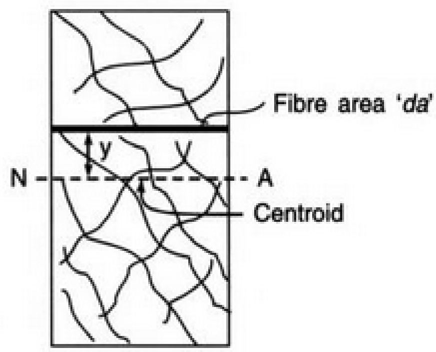


Figure 4.13 Cross section of simply supported beam.

The force acting on the fibre = bending stress × area

$$= f \cdot da$$

The total force acting over the entire section of the beam = $\int f \cdot da = \frac{E}{R}\int y \cdot da \qquad \left[\because \quad f = \frac{E}{R} y\right]$

We know that this total force is the sum of total tensile force (T) in the bottom zone and total compressive force (C) in the zone above the neutral axis. As no external force is applied over the cross section (T = C or T − C = 0), that would only mean

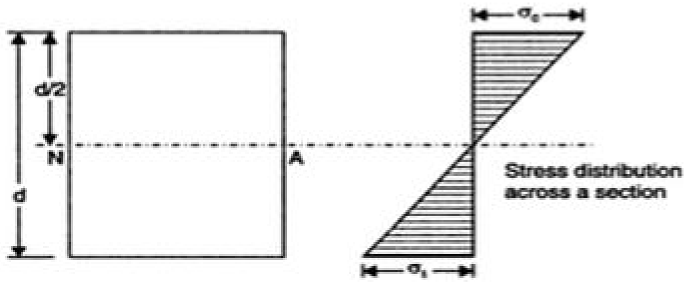
$$\frac{E}{R}\int y \cdot da = 0$$

or $\int y \times dA = 0$ $\qquad \left(\text{as } \frac{E}{R} \text{ cannot be zero}\right)$

Now $y \times dA$ represents the moment of area dA about neutral axis. Hence $\int y \times dA$ represents the moment of entire area of the section about neutral axis. But we know that moment of any area about an axis passing through its centroid, is also equal to zero. Hence neutral axis coincides with the centroidal axis. Thus the centroidal axis of a section gives the position of neutral axis.

BENDING STRESS IN SYMMETRIC SECTION

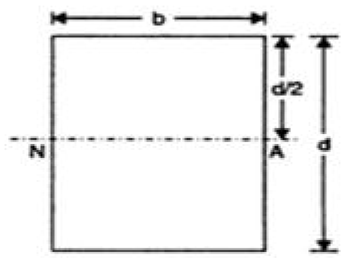
The neutral axis (N.A.) of a symmetrical section (such as circular, rectangular or square lies at a distance of $d/2$ from the outermost layer of the section where d is the diameter (for a circular section) or depth (for a rectangular or a square section). There is no stress at the neutral axis. But the stress at a point is directly proportional to its distance from the neutra axis. The maximum stress takes place at the outermost layer. For a simply supported beam there is a compressive stress above the neutral axis and a tensile stress below it. If we plo these stresses, we will get a figure as shown in Fig. 2.53.



SECTION MODULUS OF VARIOUS TYPES OF BEAMS

1. ***Rectangular Section***

Moment of inertia of a rectangular section about an axis through its C.G. (or through N.A.) is given by,



$$I = \frac{bd^3}{12}$$

Distance of outermost layer from N.A. is given by,

$$y_{max} = \frac{d}{2}$$

$\therefore$ **Section modulus is given by,**

$$Z = \frac{I}{y_{max}} = \frac{bd^3}{12 \times \left(\frac{d}{2}\right)} = \frac{bd^3}{12} \times \frac{2}{d} = \frac{bd^2}{6} \qquad \text{...(2.7)}$$

2. ***Hollow Rectangular Section***

Here

$$I = \frac{BD^3}{12} - \frac{bd^2}{12}$$

$$= \frac{1}{12}\,[BD^3 - bd^3]$$

$$y_{max} = \frac{D}{2}$$

$\therefore$

$$Z = \frac{I}{y_{max}}$$

$$= \frac{\frac{1}{12}[BD^3 - bd^3]}{\left(\frac{D}{2}\right)}$$

$$= \frac{1}{6D}\,[BD^3 - bd^3] \qquad \text{...(2.8)}$$

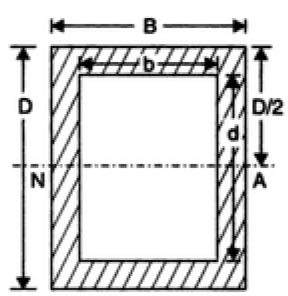


Fig. 2.55 (b)

3. *Circular Section*

For a circular section,

$$I = \frac{\pi}{64} d^4 \quad \text{and} \quad y_{max} = \frac{d}{2}$$

$$\therefore \qquad Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} d^4}{\left(\frac{d}{2}\right)} = \frac{\pi}{32} d^3 \qquad \text{...(2.9)}$$

4. *Hollow Circular Section*

Here
$$I = \frac{\pi}{64} [D^4 - d^4]$$

and
$$y_{max} = \frac{D}{2}$$

$$\therefore \qquad Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} [D^4 - d^4]}{\left(\frac{D}{2}\right)}$$

$$= \frac{\pi}{32D} [D^4 - d^4] \qquad \text{...(2.10)}$$

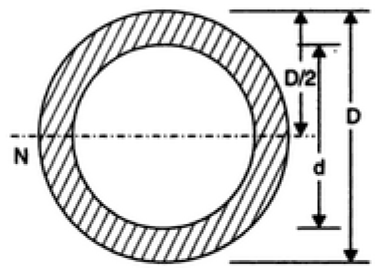


Fig. 2.55 (*c*)

STRENGTH OF A SECTION

The strength of a section means the moment of resistance offered by the section and moment of resistance is given by,

$$M = \sigma \times Z \qquad \left(\because \frac{M}{I} = \frac{\sigma}{y} \text{ or } M = \frac{\sigma}{y} \times I = \sigma \times Z \text{ where } Z = \frac{I}{y}\right)$$

where M = Moment of resistance

σ = Bending stress, and

Z = Section modulus.

For a given value of allowable stress, the moment of resistance depends upon the section modulus. The section modulus, therefore, represents the strength of the section. Greater the value of *section modulus*, stronger will be the section.

The bending stress at any point in any beam section is proportional to its distance from the neutral axis. Hence the maximum tensile and compressive stresses in a beam section are proportional to the distances of the most distant tensile and compressive fibres from the neutral axis. Hence for the purposes of economy and weight reduction the material should be concentrated as much as possible at the greatest distance from the neutral axis. This idea is put into practice, by providing beams of I-section, where the flanges alone with-stand almost all the bending stress.

We know the relation :

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad \sigma = \frac{M}{I} \times y = \frac{M}{\left(\frac{I}{y}\right)} = \frac{M}{Z}$$

where Z = Section modulus.

For a given cross-section, the maximum stress to which the section is subjected due to a given bending moment depends upon the section modulus of the section. If the section modulus is small, then the stress will be more. There are some cases where an increase in the sectional area does not result in a decrease in stress. It may so happen that in some cases a slight increase in the area may result in a decrease in section modulus which result in an increase of stress to resist the same bending moment.

Lect-22
Bending stress of beams of two materials, Composite beams.

Flitch beam

Beams that are made of more than one material are called *composite beams*. The common examples are (*a*) bimetallic beams (*b*) sandwich beams, (*c*) flitched beams, and (*d*) reinforced concrete beams, as shown in Fig. 10.26. Such composite beams can be analysed by the same bending theory because the assumption that cross-sections that are plane before bending remain plane after bending is valid in pure bending regardless of the material. The strain distribution along the depth of such a beam is linear. However, such structures are statically indeterminate, and *the position of the neutral axis is not the centroid of the section*. The criterion of *strain compatibility* has to be used, *i.e.* strain in the two materials, at a given vertical distance from the N.A., has to be the same.

If both the materials are rigidly joined together, they will behave like a unit piece and the bending will take place about the *combined axis*. On the other hand, if both the materials have been simply placed one above the other, they will bend about their respective geometrical axes. However, in both the cases, the total moment of resistance will be equal to the sum of moments of resistance of individual sections.

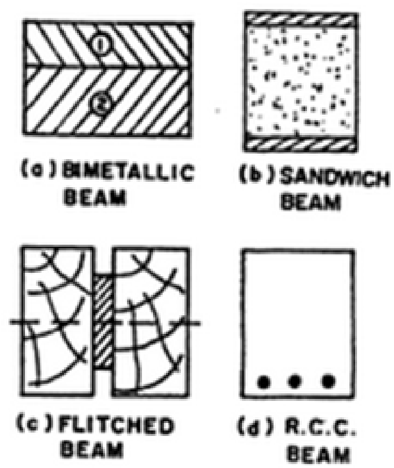


FIG. 10.26. COMPOSITE SECTIONS

(a) Symmetrical Section

Let us take the examples of two timber pieces strengthened by a steep strip sandwiched between them (Fig. 10.27 *a*). Such a beam is commonly known as a *flitched beam*.

Let D be the depth of timber planks and d be the depth of steel strip. Similarly, let B be the total width of timber section and b be the width of steel strip. Let us use suffix 1 for timber and 2 for steel. Since the steel strip is symmetrically placed, the common axis of bending will remain the same as that of the timber.

From Statics, $M = M_1 + M_2$ ...(1)

Fig. 10.27 (*b*) shows the strain distribution across the depth. At any section distant *y* from the N.A., the strain in both the materials will remain the same (strain compatibility condition), since they are in contact.

Hence $e_1 = e_2$ or $\frac{f_1}{E_1} = \frac{f_2}{E_2}$

$\therefore$ $f_2 = f_1 . \frac{E_2}{E_1} = m f_1$ ...(2)

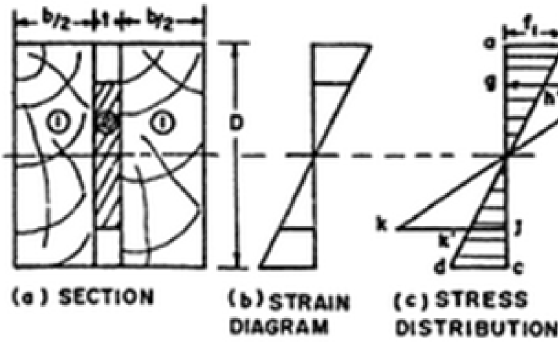


FIG. 10.27. ANALYSIS OF A FLITCHED BEAM.

where $m = \frac{E_2}{E_1}$ is known as the *modular ratio*. Since E_2 is much more than E_1, m is much greater than 1. Hence the bending stress f_2 will be much greater than f_1. The bending stress diagram is shown in Fig. 10.27(*c*) in which $f_2 = gh = m . gh' = m f_1$.

The above relation can also be obtained by considering the fact that the radius of curvature at any level will be the same for both the materials. Thus,

or $$\frac{E_1 y}{f_1} = \frac{E_2 y}{f_2}$$

or $$f_2 = \frac{E_2}{E_1} . f_1 = m f_1$$

Thus, if f_1 is given, f_2 can be found, and *vice-versa*.

If the geometry of the section is given, Z_1 and Z_2 can be calculated.

Hence $$M_1 = f_1 Z_1 \text{ and } M_2 = f_2 Z_2 \quad ...(3)$$

$\therefore$ $$\text{Total } M = M_1 + M_2 = f_1 Z_1 + f_2 Z_2$$

Thus, the total moment of resistance can be calculated.

If, however, it is required to find the *stresses* induced in the section corresponding to a given bending moment, we have :

$$R_1 = R_2 \text{ or } \frac{E_1 I_1}{M_1} = \frac{E_2 I_2}{M_2}$$

$\therefore$ $$\frac{M_1}{M_2} = \frac{E_1 I_1}{E_2 I_2} \quad ...(4)$$

From (1) and (4), M_1 and M_2 can be found. Hence the stresses f_1 and f_2 can be found by using Eq. (3).

(*b*) Unsymmetrical Section

Let us now take the example of unsymmetrical section, consisting of strip of width b and depth d_1, of material 1 and strip of width b and depth d_2 of material 2. The first step will be to find the position of *new axis of bending*. This be best done by drawing what is known as the *equivalent section* (Fig 10.28 *b*).

Fig 10.28(*a*) shows the original section. Let us find the equivalent section of plate 2 in terms of plate 1.

For the original section of plate 2, we have $M_2 = f_2 Z_2$

If M_1' and Z_1' are the moment of resistance and section modulus of equivalent section of plate 2, we have $M_1' = f_1 Z_1'$.

If M_1' has to be equal to M_2, we have

$$f_2 Z_2 = f_1 Z_1'$$

or
$$Z_1' = \frac{f_2}{f_1} Z_2 = m\, Z_2$$

Hence $\quad b' = m\, b$

(Since Z_1' and Z_2 are proportional to b)

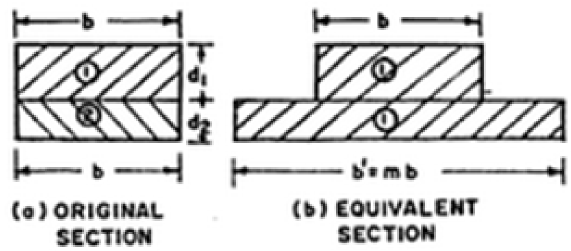


FIG 10.28. COMPOSITE SECTION

Sandwich Beams

A sandwich beam consists of

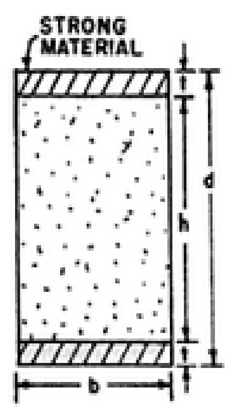
(*i*) two thin layers of strong material, called *faces*, placed at top and bott

(*ii*) thick *core*, consisting of light weight, low strength material. The core sin as a *filler* or *spacer*. Sandwich construction is used where light weight combined strength and high stiffness are needed.

Sandwich beams can be analysed by two method :

(*i*) *Method 1* : Same as described for composite beam or flitched beam.

(*ii*) *Method 2* : An approximate theory for bending can be used, based on the assumption that the *faces* carry all the longitudinal bending stresses. Such an approximation is valid, specially when the core have very low modulus of elasticity in comparison to that of the faces.



If I_f is the moment of inertia of the faces about the bending axes, we have $\quad I_f = \frac{1}{12} b\,(d^3 - h^3)$

Hence $\quad f_f = \frac{M}{I_f} \cdot \frac{d}{2} = \frac{M\,d}{2\,I_f} \quad$ ...(10.18)

where f_f is the bending stress at the outermost edge of the beam.

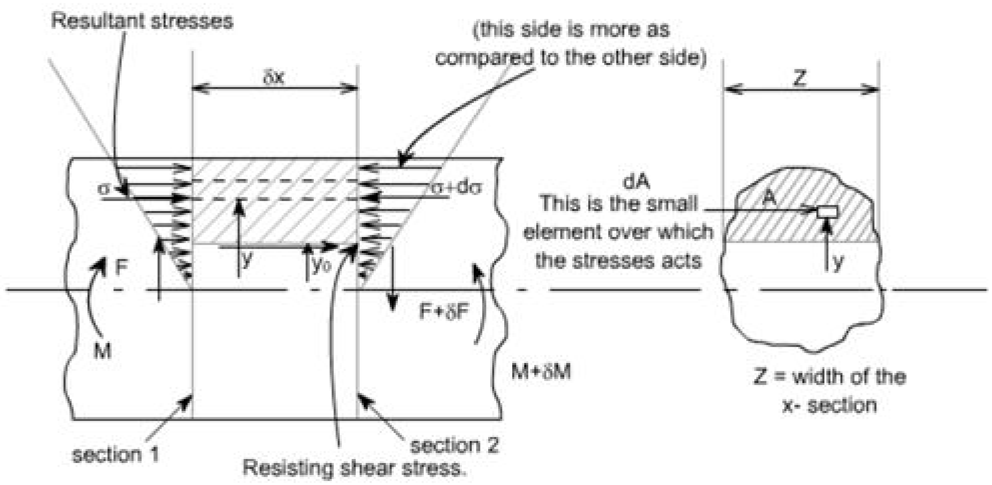
Lect-23
Shear stresses in the beam.

Shear stresses in the beam

In the last section, we have seen that when a part of a beam is subjected to a constant bending moment and zero shear force, then there will be only bending stresses in the beam. The shear stress will be zero as shear stress is equal to shear force divided by the area. As shear force is zero, the shear stress will also be zero.

But in actual practice, a beam is subjected to a bending moment which varies from section to section. Also the shear force acting on the beam is not zero. It also varies from section to section. Due to these shear forces, the beam will be subjected to shear stresses. These shear stresses will be acting across transverse sections of the beam. These transverse shear stresses will produce a complimentary horizontal shear stresses, which will be acting on longitudinal layers of the beam. Hence beam will also be subjected to shear stresses. In this section, the distribution of the shear stress across the various sections (such as Rectangular section, Circular section, I-section, T-sections etc.) will be determined.

SHEAR STRESS IN A SECTION



ASSUMPTIONS

1. Stress is uniform across the width (i.e. parallel to the neutral axis)

2. The presence of the shear stress does not affect the distribution of normal bending stresses.

It may be noted that the assumption no.2 cannot be rigidly true as the existence of shear stress will cause a distortion of transverse planes, which will no longer remain plane.

It may be noted that the assumption no.2 cannot be rigidly true as the existence of shear stress will cause a distortion of transverse planes, which will no longer remain plane.

In the above figure let us consider the two transverse sections which are at a distance □ **δx'** apart. The shearing forces and bending moments being **F, F + δF** and **M, M + δM** respectively. Now due to the shear stress on transverse planes there will be a complementary shear stress on longitudinal planes parallel to the neutral axis.

Let **τ**□be the value of the complementary shear stress (and hence the transverse shear stress) at a distance □**Y'$_0$** from the neutral axis. **Z** is the width of the x-section at this position

A is area of cross-section cut-off by a line parallel to the neutral axis.

$\bar{y}$ = distance of the centroid of Area from the neutral axis.

Let **σ , σ + dσ** are the normal stresses on an element of area **δA** at the two transverse sections, then there is a difference of longitudinal forces equal to (**dσ . δA)** , and this quantity summed over the area A is in equilibrium with the transverse shear stress **τ** on the longitudinal plane of area z **δx** .

i.e $\tau . z\delta x = \int d\sigma . dA$

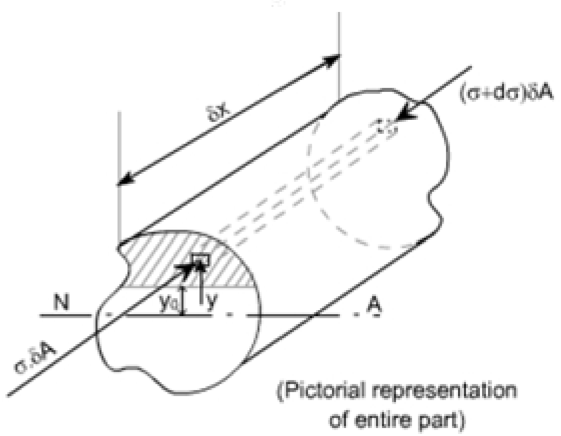
from the bending theory equation

$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M . y}{I}$$

$$\sigma + d\sigma = \frac{(M + \delta M) . y}{I}$$

Thus $$d\sigma = \frac{\delta M . y}{I}$$



$$d\sigma = \frac{\delta M . y}{I}$$

$$\tau . z\delta x = \int d\sigma . dA$$

$$= \int \frac{\delta M \ . \ y . \delta A}{I}$$

$$\tau . z\delta x = \frac{\delta M}{I} \int y . \delta A$$

But $F = \frac{\delta M}{\delta x}$

i.e. $\tau = \frac{F}{I . z} \int y . \delta A$

But from definition, $\int y . dA = A\bar{y}$

$\int y . dA$ is the first moment of area of the shaded portion

and $\bar{y}$ = centroid of the area'A'

Hence

So substituting

$$\tau = \frac{F . A\bar{y}}{I . z}$$

Where z' is the actual width of the section at the position where τ ' is being calculated and I is the total moment of inertia about the neutral axis.

Lect-24
Shear stress devlope in different section, , solve related problem

SHEAR STRESS DISTRIBUTION IN SOME SECTION

The following are the important sections over which the shear stress distribution is to be obtained :

1. Rectangular Section,
2. Circular Section,
3. I-Section,
4. T-Sections, and
5. Miscellaneous Sections.

1.RECTANGULAR SECTION :

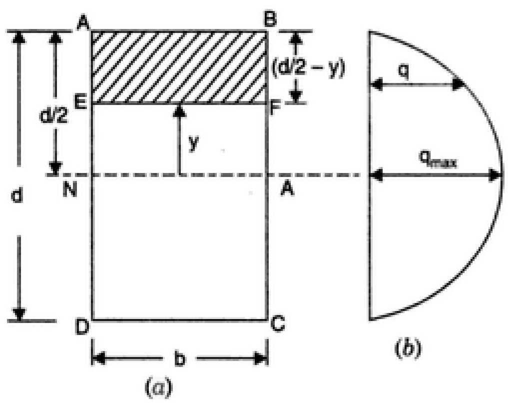
a rectangular section of a beam of width b and depth d. Let F is the shear force acting at the section. Consider a level EF at a distance y from the neutral axis.

The shear stress at this level is given by equation (2.11) as

$$\tau = F \cdot \frac{A\bar{y}}{b \times I}$$

where A = Area of the section above y (*i.e.*, shaded area $ABFE$)

$$= \left(\frac{d}{2} - y\right) \times b$$



$\bar{y}$ = Distance of the C.G. of area A from neutral axis

$$= y + \frac{1}{2}\left(\frac{d}{2} - y\right) = y + \frac{d}{4} - \frac{y}{2} = \frac{y}{2} + \frac{d}{4} = \frac{1}{2}\left(y + \frac{d}{2}\right)$$

b = Actual width of the section at the level EF

I = M.O.I. of the whole section about N.A.

Substituting these values in the above equation, we get

$$\tau = \frac{F.\left(\frac{d}{2} - y\right) \times b \times \frac{1}{2}\left(y + \frac{d}{2}\right)}{b \times I}$$

$$= \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right) \qquad \text{...(2.12)}$$

From equation (2.12), we see that τ increases as y decreases. Also the variation of τ with respect to y is a parabola. Fig. 2.82 (b) shows the variation of shear stress across the section.

At the top edge, $y = \frac{d}{2}$ and hence

$$\tau = \frac{F}{2I}\left[\frac{d^2}{4} - \left(\frac{d}{2}\right)^2\right] = \frac{F}{2I} \times 0 = 0$$

At the neutral axis, $y = 0$ and hence

$$\tau = \frac{F}{2I}\left(\frac{d^2}{4} - 0\right) = \frac{F}{2I} \times \frac{d^2}{4}$$

$$= \frac{Fd^2}{8I} = \frac{Fd}{8 \times \frac{bd^3}{12}} \qquad \left(\because I = \frac{bd^3}{12}\right)$$

$$= \frac{12}{8} - \frac{F}{bd} = 1.5\,\frac{F}{bd} \qquad \text{...(i)}$$

Now average shear stress, $\tau_{avg} = \frac{\text{Shear force}}{\text{Area of section}} = \frac{F}{b \times d}$.

Substituting the above value in equation (i), we get

$$\tau = 1.5 \times \tau_{avg} \qquad \text{...(2.13)}$$

Equation (2.13) gives the shear stress at the neutral axis where $y = 0$. This stress is also the maximum shear stress.

$$\therefore \qquad \tau_{max} = 1.5\tau_{avg} \qquad \text{...(2.14)}$$

From equation (2.11), $\tau = \frac{A\bar{y}}{Ib}$. In this equation the value of $A\bar{y}$ can also be calculated as given below :

$$A\bar{y} = \text{Moment of shaded area of Fig. 2.82 } (a) \text{ about N.A.}$$

Consider a strip of thickness dy at a distance y from N.A. Let dA is the area of this strip.

Then dA = Area of strip = $b \times dy$

Moment of the area dA about N.A.

$$= dA.\,y \quad \text{or} \quad y \times dA$$

$$= y \times bdy \qquad (\because \; dA = b \times dy)$$

The moment of the shaded area about N.A. is obtained by integrating the above equation between the limits y to $\frac{d}{2}$.

∴ Moment of shaded area about N.A.

∴ Moment of shaded area about N.A.

$$= \int_y^{d/2} y \times b \times dy$$

$$= b \int_y^{d/2} y \times dy \qquad \text{(as } b \text{ is constant)}$$

$$= b\left[\frac{y^2}{2}\right]_y^{d/2} = \frac{b}{2}\left[\left(\frac{d}{2}\right)^2 - y^2\right] = \frac{b}{2}\left[\frac{d^2}{4} - y^2\right]$$

But moment of shaded area about N.A. is also equal to $A\bar{y}$

$$\therefore \qquad A\bar{y} = \frac{b}{2}\left[\frac{d^2}{4} - y^2\right]$$

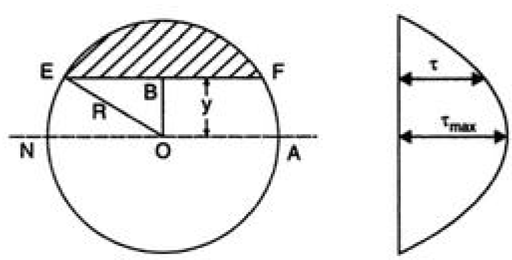
Substituting the value of $A\bar{y}$ in equation (2.11), we get

$$\tau = \frac{F \times \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)}{I \times b} = \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right)$$

CIRCULAR SECTION :

a circular section of a beam. Let R is the radius of the circular section of F is the shear force acting on the section. Consider a level EF at a distance y from the neutral axis.

The shear stress at this level is given by equation (2.11) as



(a) (b)

$$\tau = \frac{F \times A \times \bar{y}}{I \times b}$$

where $A\bar{y}$ = Moment of the shaded area about the neutral axis (N.A.)

I = Moment of inertia of the whole circular section

b = Width of the beam at the level EF.

Consider a strip of thickness dy at a distance y from N.A. Let dA is the area of strip.

Then $$dA = b \times dy = EF \times dy \qquad (\because \ b = EF)$$

$$= 2 \times EB \times dy \qquad (\because \ EF = 2 \times EB)$$

$$= 2 \times \sqrt{R^2 - y^2} \times dy$$

($\because$ In right angled triangle OEB, side $EB = \sqrt{R^2 - y^2}$)

Moment of this area dA about N.A.

$$= y \times dA$$

$$= y \times 2\sqrt{R^2 - y^2} \times dy \qquad (\because dA = 2\sqrt{R^2 - y^2}\, dy)$$

$$= 2y\sqrt{R^2 - y^2}\ dy.$$

Moment of the whole shaded area about the N.A. is obtained by integrating the above equation between the limits y and R

$$\therefore \qquad A\bar{y} = \int_y^R 2y\sqrt{R^2 - y^2}\ dy$$

$$= -\int_y^R (-2y)\sqrt{R^2 - y^2}\ dy.$$

Now $(-2y)$ is the differential of $(R^2 - y^2)$. Hence, the integration of the above equation becomes as

$$A\bar{y} = -\left[\frac{(R^2 - y^2)^{3/2}}{3/2}\right]_y^R$$

$$= -\frac{2}{3}\left[(R^2 - R^2)^{3/2} - (R^2 - y^2)^{3/2}\right]$$

$$= -\frac{2}{3}\left[0 - (R^2 - y^2)^{3/2}\right] = \frac{2}{3}(R^2 - y^2)^{3/2}$$

Substituting the value of $A\bar{y}$ in equation (*i*), we get

$$\tau = \frac{F \times \frac{2}{3}(R^2 - y^2)^{3/2}}{I \times b}$$

But $\qquad b = EF = 2 \times EB = 2 \times \sqrt{R^2 - y^2}$

Substituting this value of b in the above equation, we get

$$\tau = \frac{\frac{2}{3}F(R^2 - y^2)^{3/2}}{I \times 2\sqrt{R^2 - y^2}} = \frac{F}{EI}(R^2 - y^2) \qquad \text{...(2.15)}$$

Equation (2.15) shows that shear stress distribution across a circular section is parabolic. Also it is clear from this equation that with the increase of y, the shear stress decreases. At $y = R$, the shear stress, $\tau = 0$. Hence, shear stress will be maximum when $y = 0$ *i.e.*, at the neutral axis.

$\therefore$ At $y = 0$ *i.e.*, at the neutral axis, the shear stress is maximum and is given by

$$\tau_{max} = \frac{F}{3I}R^2$$

But $\qquad I = \frac{\pi}{64}D^4 = \frac{\pi}{64} \times (2R)^4 \qquad (\because\ D = 2R)$

$$= \frac{\pi}{4}R^4$$

$$\therefore \qquad \tau_{max} = \frac{F \times R^2}{3 \times \frac{\pi}{4}R^4} = \frac{4}{3} \times \frac{F}{\pi R^2}$$

But average shear stress,

$$\tau_{avg} = \frac{\text{Shear force}}{\text{Area of circular section}} = \frac{F}{\pi R^2}$$

Hence equation (2.16) becomes as,

$$\tau_{max} = \frac{4}{3} \times \tau_{avg}$$

I-section

Consider a I section

Let B = Overall width of the section,

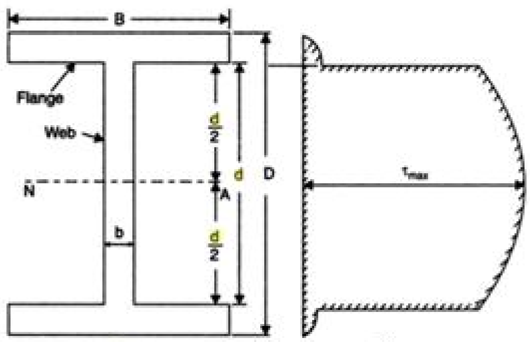
D = Overall depth of the section,

b = Thickness of the web, and

d = Depth of web.

The shear stress at a distance y from the N.A., is given by equation (2.11) as

$$\tau = F \times \frac{A\bar{y}}{I \times b}.$$



In this case the shear stress distribution in the web and shear stress distribution in the flange are to be calculated separately. Let us first calculate the shear stress distribution in the flange.

(*i*) *Shear stress distribution in the flange*

Consider a section at a distance y from N.A. in the flange as shown in Fig. 2.87 (*c*).

Width of the section = B

Shaded area of flange, $A = B\left(\frac{D}{2} - y\right)$

Distance of the C.G. of the shaded area from neutral axis is given as

$$\bar{y} = y + \frac{1}{2}\left(\frac{D}{2} - y\right)$$

$$= y + \frac{D}{4} - \frac{y}{2}$$

$$= \frac{D}{4} + \frac{y}{2} = \frac{1}{2}\left(\frac{D}{2} + y\right)$$

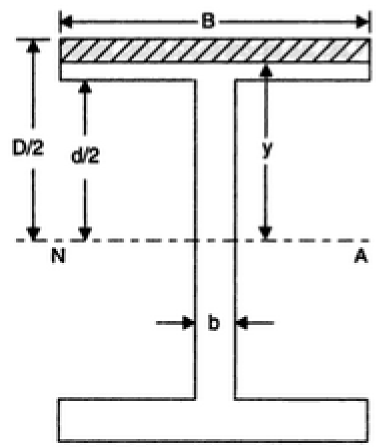


Fig. 2.88 (*c*)

Hence shear stress in the flange becomes,

$$\tau = \frac{F \times A\bar{y}}{I \times B} \qquad (\because \text{ Here width} = B)$$

$$= \frac{F \times B\left(\frac{D}{2} - y\right) \times \frac{1}{2}\left(\frac{D}{2} + y\right)}{I \times B}$$

$$= \frac{F}{2I}\left[\left(\frac{D}{2}\right)^2 - y^2\right]$$

$$= \frac{F}{2I}\left(\frac{D^2}{2} - y^2\right) \qquad ...(2.18)$$

Hence, the variation of shear stress (τ) with respect to y in the flange is parabolic. It is also clear from equation (2.18) that with the increase of y, shear stress decreases.

(a) For the upper edge of the flange,

$$y = \frac{D}{2}$$

Hence shear stress, $$\tau = \frac{F}{2I}\left[\frac{D^2}{4} - \left(\frac{D}{2}\right)^2\right] = 0.$$

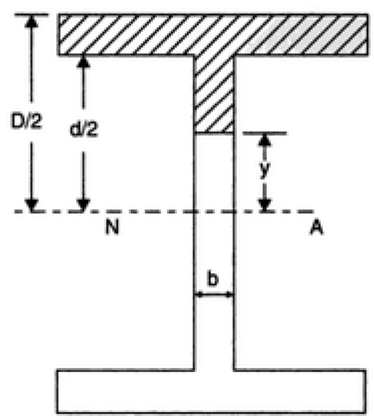
(b) For the lower edge of the flange,

$$y = \frac{d}{2}$$

Hence $$\tau = \frac{F}{2I}\left[\frac{D^2}{4} - \left(\frac{d}{2}\right)^2\right] = \frac{F}{2I}\left(\frac{D^2}{4} - \frac{d^2}{4}\right)$$

$$= \frac{F}{8I}(D^2 - d^2) \qquad ...(2.19)$$

(ii) *Shear stress distribution in the web*



Consider a section at a distance y in the web from the N.A. as shown in Fig. 2.89.

Width of the section = b.

Here $A\bar{y}$ is made up of two parts *i.e.,* moment of the flange area about N.A. plus moment of the shaded area of the web about the N.A.

$\therefore A\bar{y}$ = Moment of the flange area about N.A. + moment of the shaded area of web about N.A.

$$= B\left(\frac{D}{2} - \frac{d}{2}\right) \times \frac{1}{2}\left(\frac{D}{2} + \frac{d}{2}\right) + b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(\frac{d}{2} + y\right)$$

$$= \frac{B}{8}(D^2 - d^2) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Hence the shear stress in the web becomes as

$$\tau = \frac{F \times A\bar{y}}{I \times b} = \frac{F}{I \times b} \times \left[\frac{B}{8}(D^2 - d^2) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right] \qquad ...(2.20)$$

From equation (2.20), it is clear that variation of τ with respect to y is parabolic. Also with the increase of y, τ decreases.

At the neutral axis, $y = 0$ and hence shear stress is maximum.

$$\therefore \qquad \tau_{max} = \frac{F}{I \times b}\left[\frac{B}{8}(D^2 - d^2) + \frac{b}{2} \times \frac{d^2}{4}\right]$$

$$= \frac{F}{I \times b}\left[\frac{B(D^2 - d^2)}{8} + \frac{bd^2}{8}\right]$$

At the junction of top of the web and bottom of flange,

$$y = \frac{d}{2}$$

Hence shear stress is given by,

$$\tau = \frac{F}{I \times b}\left[\frac{B}{8}(D^2 - d^2) + \frac{b}{2}\left(\frac{d^2}{4} - \left(\frac{d}{2}\right)^2\right)\right]$$

$$= \frac{F \times B \times (D^2 - d^2)}{8I \times b} \qquad ...(2.22)$$

The shear stress distribution for the web and flange is shown in Fig. 2.88 (*b*). The shear stress at the junction of the flange and the web changes abruptly. The equation (2.19) gives the stress at the junction of the flange and the web when stress distribution is considered in the flange. But equation (2.22) gives the stress at the junction when stress distribution is considered in the web. From these two equations it is clear that the stress at the junction changes abruptly from $\frac{F}{8I}(D^2 - d^2)$ to $\frac{B}{b} \times \frac{F}{8I}(D^2 - d^2)$.

Lect-25

Differential equation of the elastic line

Introduction:

In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods

by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

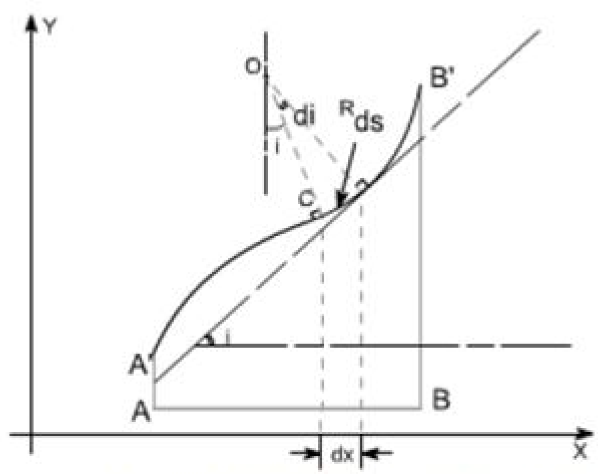
Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

2. The curvature is always small.

3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i \approx i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or } \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force,bending moment and deflection of the beam may be obtained as

Differentiating the equation as derived

$$\frac{dM}{dx}=EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx}=F$$

Thus,

$$F=EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Lect-26

Slope and deflection of beams by integration method, solve related problem

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods. The one of the method for finding the deflection of the beam is the direct integration method, i the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

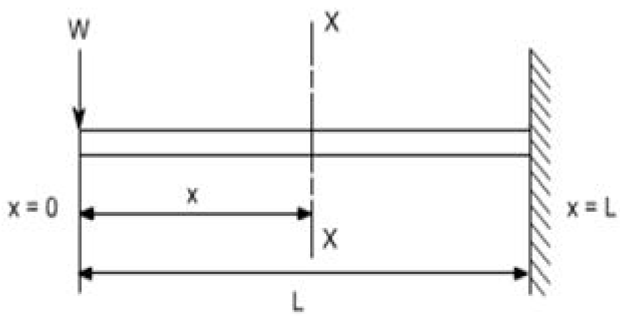
on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

<u>**Case 1:** Cantilever Beam with Concentrated Load at the end:-</u> A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

$$\text{Therefore } M|_{x-x} = -W.x$$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

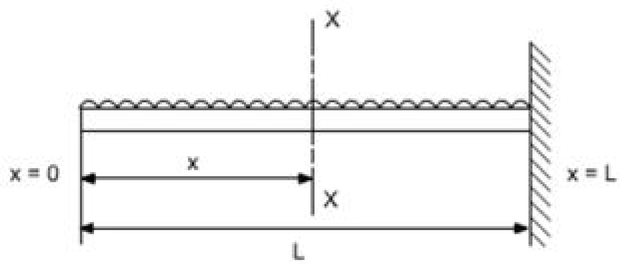
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$y_{max} = -\frac{WL^3}{3EI}$$

$$(Slope)_{max}m = +\frac{WL^2}{2EI}$$

Case 2: A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



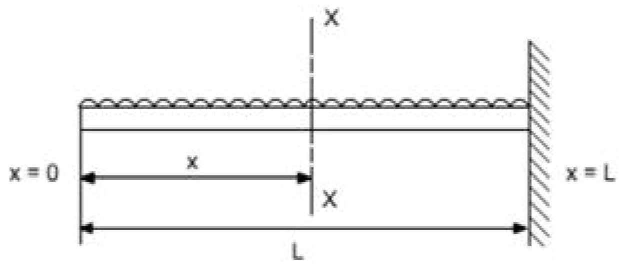
$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$y_{max} = -\frac{WL^3}{3EI}$$

$$(Slope)_{max}m = +\frac{WL^2}{2EI}$$

Case 2: A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int\frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0

2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

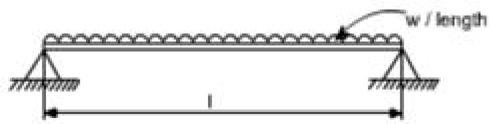
Thus, $y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$

So y_{maxm} will be at $x = 0$

$$\boxed{y_{maxm} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{maxm} = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads:- In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



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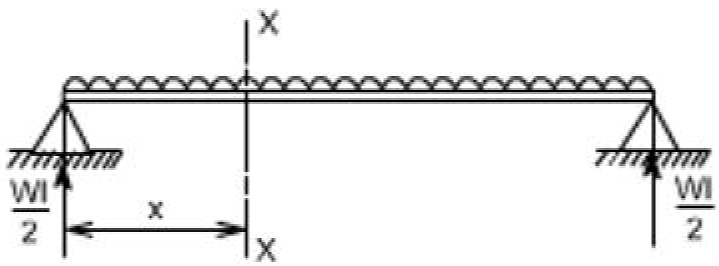
In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----(1)}$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

Then $$y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

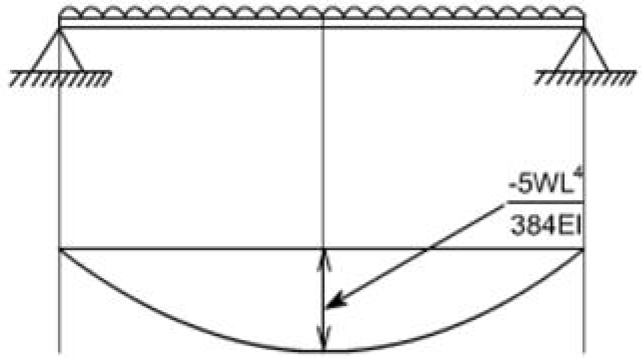
(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2.

The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

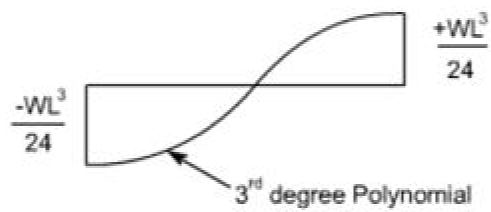


Deflection (y)

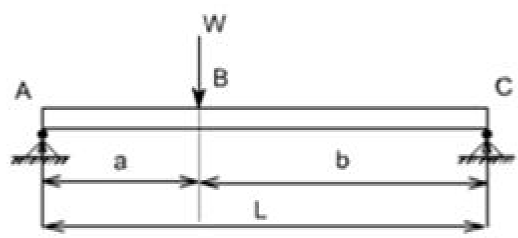
$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Slope (dy/dx)

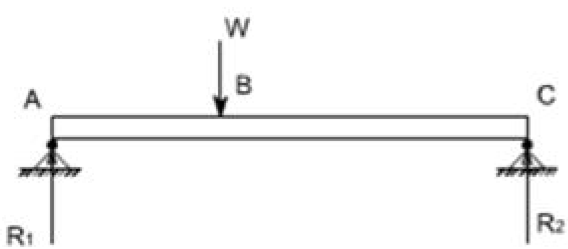
$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \leq x \leq a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \leq x \leq l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1 x$$

$$EI\frac{d^2y}{dx^2} = R_1 x - W(x - a)$$

These two equations can be integrated in the usual way to find □y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at $x = a$. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at $x = 0$; $y = 0$ in the portion AB i.e. $0 \le x \le a$

(b) at $x = l$; $y = 0$ in the portion BC i.e. $a \le x \le l$

(c) at $x = a$; dy/dx, the slope is same for both portion

(d) at $x = a$; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{-------} (4)$$

Integrating agian equation (3) and (4) we get

$$Ely = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{------}(5)$$

$$Ely = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{------}(6)$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{----}\mathbf{for\ 0 \le x \le a}\text{-----}(7)$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{For } \mathbf{for\ a \le x \le l}\text{-----}(8)$$

so either of the equation (7) or (8) may be used to find the deflection at x = a
hence substituting x = a in either of the equation we get

$$Y\Big|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

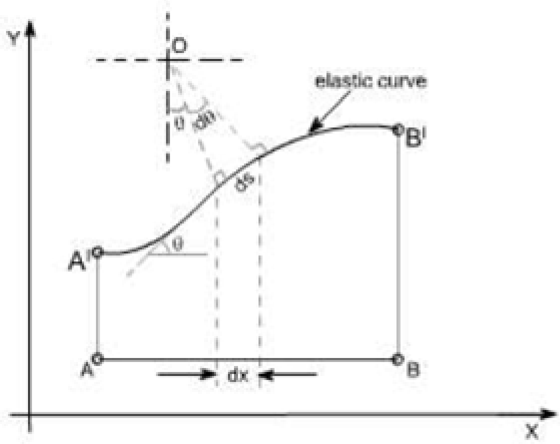
$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

Lect-27
Slope and deflection area moment method, , solve related problem

Slope and deflection area moment method

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that dθ is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.

We can assume,

ds = dx [since the curvature is small]

hence, R dθ = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

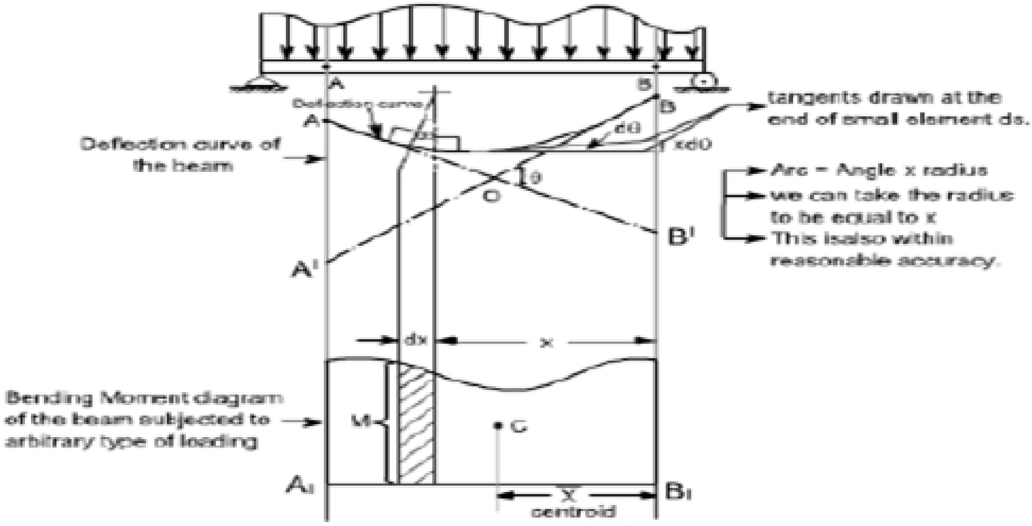
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle, slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$, hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \quad \text{-----} (1)$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be dθ. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle θ between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or }\theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI}\times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x dθ [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle θ]

Hence the total distance B'B becomes $\delta=\int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of dθ = M dx / EI as derived earlier

$$\delta = \int_A^B x\frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$ [This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Theorem II:

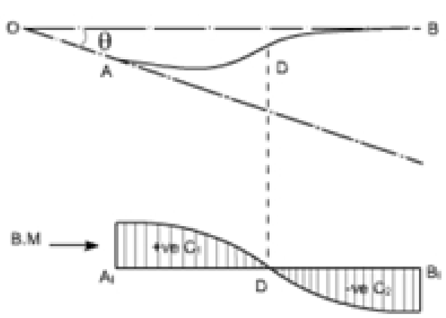
Deflection of point B' relative to point A $= \frac{1}{EI}\times\left\{\begin{matrix}\text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram}\end{matrix}\right\}$

Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $$\boxed{\delta_A = \frac{1}{EI}A\bar{x}}$$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and ve portions with centroids $C_1$and C_2. Then to find an angle θ between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Lect-28
Eccentric loading of a short strut

Introduction:

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stres of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stre in compression is reached. The buckling occurs owing to one the following reasons.

(a). the strut may not be perfectly straight initially.

(b). the load may not be applied exactly along the axis of the Strut.

(c). one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

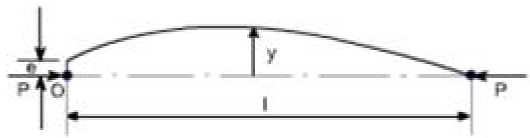
Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

Eccentric loading of a short strut

Let e' be the eccentricity of the applied end load, and measuring y from the line of action of the load.



Then $E\,I\,\dfrac{d^2y}{dx^2} = -\,P\,y$

or $(D^2 + n^2)\,y = 0$ where $n^2 = P / EI$

Therefore $y_{general} = y_{complementary}$

$= A\sin nx + B\cos nx$

applying the boundary conditions then we can determine the constants i.e.

at x = 0 ; y = e thus B = e

at x = l / 2 ; dy / dx = 0

Therefore

$$A\cos\frac{nl}{2} - B\sin\frac{nl}{2} = 0$$

$$A\cos\frac{nl}{2} = B\sin\frac{nl}{2}$$

$$A = B\tan\frac{nl}{2}$$

$$A = e\tan\frac{nl}{2}$$

Hence the complete solution becomes

$y = A\sin(nx) + B\cos(nx)$

substituting the values of A and B we get

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

Note that with an eccentric load, the strut deflects for all values of P, and not only for the critical value as was the case with an axially applied load. The deflection becomes infinite for tan (nl)/2 = ∞ i.e. nl = π giving the same crippling load $P_e = \frac{\pi^2 EI}{l^2}$. However, due to additional bending moment set up by deflection, the strut will always fail by compressive stress before Euler load is reached.

Since

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

$$y_{max^m}\Big|_{at\ x = \frac{l}{2}} = e\left[\tan\left(\frac{nl}{2}\right)\sin\frac{nl}{2} + \cos\frac{nl}{2}\right]$$

$$= e\left[\frac{\sin^2\frac{nl}{2} + \cos^2\frac{nl}{2}}{\cos\frac{nl}{2}}\right]$$

$$= e\left[\frac{1}{\cos\frac{nl}{2}}\right] = e\ \sec\frac{nl}{2}$$

Hence maximum bending moment would be

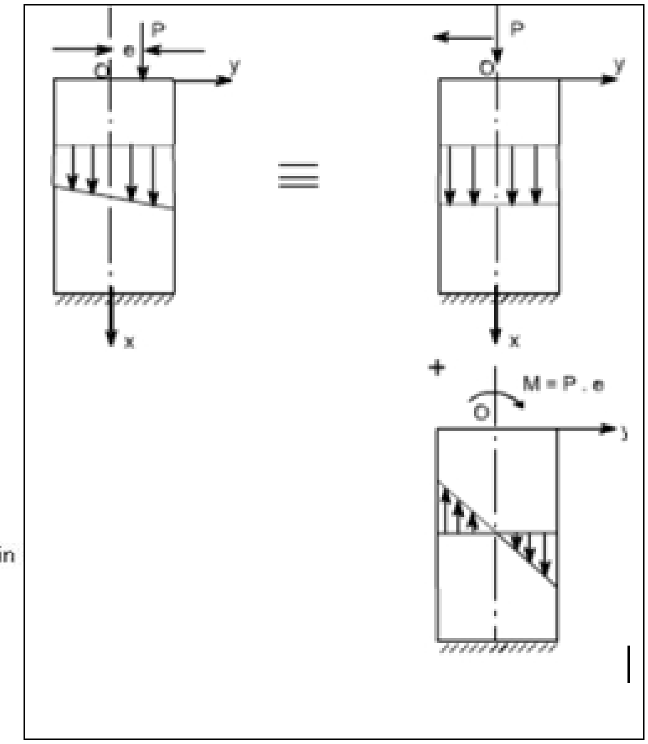
$$M_{max^m} = P\,y_{max^m}$$

$$= P\,e\ \sec\frac{nl}{2}$$

Now the maximum stress is obtained by combined and direct strain

$\sigma = \frac{P}{A} + \frac{M}{Z}$ stress due to bending $\frac{\sigma}{y} = \frac{M}{I}$;

$M = \sigma\frac{I}{y}$; $\sigma_{max} = \frac{M}{Z}$ Wher Z = I/y is section modulus



The second term is obviously due the bending action.

Consider a short strut subjected to an eccentrically applied compressive force P at its upper end. If such a strut is comparatively short and stiff, the deflection due to bending action of the eccentric load will be neglible compared with eccentricity e' and the principal of super-imposition applies.

If the strut is assumed to have a plane of symmetry (the xy - plane) and the load P lies in this plane at the distance e' from the centroidal axis ox.

Then such a loading may be replaced by its statically equivalent of a centrally applied compressive force P' and a couple of moment P.e

1. The centrally applied load P produces a uniform compressive $\sigma_1 = \frac{P}{A}$ stress over each cross-section as shown by the stress diagram.

2. The end moment M' produces a linearly varying bending stress $\sigma_2 = \frac{My}{I}$ as shown in the figure.

Then by super-impostion, the total compressive stress in any fibre due to combined bending and compression becomes,

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

$$\sigma = \frac{P}{A} + \frac{M}{I/y}$$

$$\boxed{\sigma = \frac{P}{A} + \frac{M}{Z}}$$

Lect-29
Eccentric loading on a long column

Eccentric loading on a long column

Rakinkine formula

Consider a long column which is subjected to eccentric load. Let

P = Load on the column
A = Area of cross-section
e = Eccentricity
k = Least radius of gyration
y_c = Distance of extreme fibre on compression side from axis of the column

The maximum intensity of compressive stress for eccentric load on column is:

$$\sigma_{max} = \text{Direct stress} + \text{Bending stress}$$
$$= \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + \frac{Pe}{Z}$$
$$= \frac{P}{A} + \frac{Pey_c}{Ak^2} \qquad \left(\because Z = \frac{Ak^2}{y_c}\right)$$
$$= \frac{P}{A}\left(1 + \frac{ey_c}{k^2}\right)$$

or

$$P = \frac{\sigma_{max} A}{\left(1 + \frac{ey_c}{k^2}\right)}$$

Let σ_c be the safe stress for the column material. Therefore, the safe load on the column with an eccentricity e is given by

$$P = \frac{\sigma_c A}{\left(1 + \frac{ey_c}{k^2}\right)}$$

Equation (6.39) gives

$$P = \frac{\sigma_c A}{1 + a\left(\frac{L}{k}\right)^2}$$

If buckling is to be included, the safe eccentric load P becomes

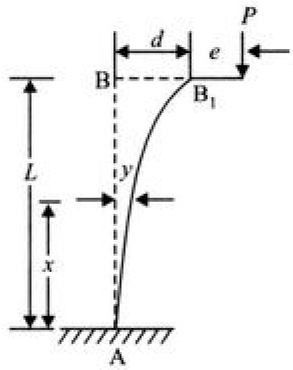
$$P = \frac{\sigma_c A}{\left(1 + \frac{ey_c}{k^2}\right)\left(1 + a\frac{L^2}{k^2}\right)}$$

EULER'S FORMULA

A column AB fixed at the end A and free at B is shown to deflect a distance d under the action of eccentric load P (Figure 6.8). The length of the beam is L. Due to this loading, beam AB takes the curved shape AB_1. The free end B occupies a new position as shown in Figure 6.8.

Consider a section at a distance x from A. Let

P = Critical load on the column
d = Deflection of B
e = Eccentricity of the load
y = Deflection of the column at x



Moment at x due the load $P = M = P(d + e - y) = P(d + e) - Py$

or
$$EI\frac{d^2y}{dx^2} = P(d + e) - Py$$

or
$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{P(d+e)}{EI}$$

The general solution of the above differential equation is:

$$y = C_1 \cos\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}}x\right) + (d + e) \tag{6.46}$$

Here C_1 and C_2 are constants of integration. It is seen that at $x = 0$, $y = 0$, hence from Eq. (6.46), we get

$$C_1 = -(d + e)$$

Differentiating Eq. (6.46), we have

$$\frac{dy}{dx} = C_1\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}x\right) \tag{6.47}$$

Again boundary conditions at $x = 0$, is given as:

$$x = 0, \frac{dy}{dx} = 0$$

Substituting these boundary conditions in Eq. (6.47), we obtain

$$0 = C_2\sqrt{\frac{P}{EI}} \tag{6.48}$$

In Eq. (6.48), C_2 must be 0 as the load P is not 0.

Now, substituting $C_1 = -(d + e)$ and $C_2 = 0$ in Eq. (6.46), we get

$$y = -(d+e)\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + (d+e)$$

Another boundary condition is at $x = L$, $y = d$. Therefore,

$$d = -(d+e)\left(\cos L\sqrt{\frac{P}{EI}}\right) + (d+e)$$

or

$$(d+e)\cos L\sqrt{\frac{P}{EI}} = e$$

or

$$(d+e) = e\sec L\sqrt{\frac{P}{EI}}$$

Now, the maximum bending moment occurs at B and is given by

$$M_{max} = P(d+e)$$

Substituting the value of $(d + e)$ from Eq. (6.49), we get

$$M_{max} = Pe\sec L\sqrt{\frac{P}{EI}}$$

The maximum compressive stress occurs at A such that,

$$\sigma_{max} = \text{Direct stress} + \text{Bending stress}$$

$$= \frac{P}{A} + \frac{M}{Z}$$

$$= \frac{P}{A} + \frac{Pe\sec L\sqrt{\frac{P}{EI}}}{Z}$$

Similarly, it can be shown that the equation of maximum bending moment for hinged and fixed ends of column are as follows:

1. When both ends of column are hinged, $M_{max} = Pe\sec\frac{L}{2}\sqrt{\frac{P}{EI}}$
2. When both ends of column are fixed, $M_{max} = Pe\sec\frac{L}{4}\sqrt{\frac{P}{EI}}$

Lect-30
Euler's column formula

ASSUMPTIONS MADE IN THE EULER'S COLUMN THEORY

The following assumptions are made in the Euler's column theory:

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column fails by buckling alone.
7. The self-weight of the column is neglected.

EXPRESSIONS FOR CRIPPLING LOAD OF DIFFERENT CASES

Both the Ends are Hinged or Pinned

Figure 10.1 shows a column of length l of uniform cross-section hinged at both the ends. Let P be the crippling load at which the column has just buckled. Consider section XX at a distance x from bottom support. Let y be the deflection at the section.

Bending moment M at the section is given by

$$M = EI\frac{d^2y}{dx^2} = -Py$$

or

$$EI\frac{d^2y}{dx^2} + Py = 0$$

or

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

The solution of above differential equation is:

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) \qquad (10.1)$$

where C_1 and C_2 are the constants of integration.

At $x = 0$, $y = 0$,

From Eq. (10.1) $\qquad C_1 = 0$

At $x = l$, $y = 0$,

From Eq. (10.1) $\qquad 0 = C_2 \sin\left(l\sqrt{\frac{P}{EI}}\right)$

or $\qquad C_2 = 0$

or $\qquad \sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$

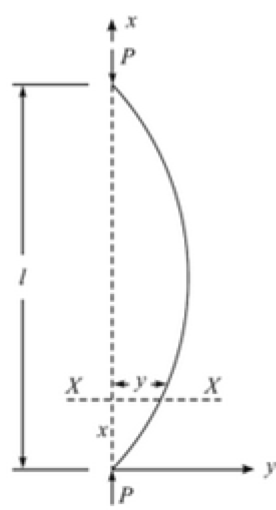


Fig. 10.1 Column with both ends hinged.

As $C_1 = 0$, and if $C_2 = 0$, then from Eq. (10.1) we will get $y = 0$. This means the bending will be zero, i.e., column will not bend, which is not true.

$$\therefore \qquad C_2 \neq 0$$

Now,

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 = \sin 0,\ \sin \pi,\ \sin 2\pi,\ \sin 3\pi,\ ...,$$

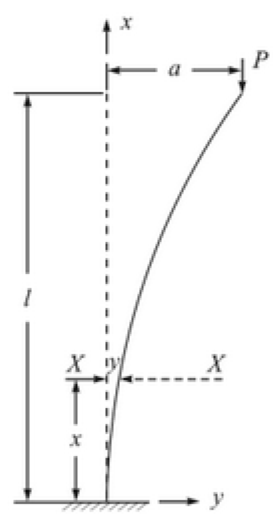
$$\therefore \qquad \left(l\sqrt{\frac{P}{EI}}\right) = 0,\ \pi,\ 2\pi,\ 3\pi, ...,$$

Consider the least practical value:

$$\left(l\sqrt{\frac{P}{EI}}\right) = \pi$$

$$P = \frac{\pi^2 EI}{l^2}$$

One End is Fixed and Other is Free



Consider a column of length l where lower end is fixed and upper being free. Let due to crippling load P the column just buckle. Let deflection at the top end is 'a'. Consider a section XX at a distance x from the lower end.

The bending moment at the section XX is:

$$EI\frac{d^2y}{dx^2} = P(a - y)$$

where y is the deflection at distance x.

Now,

$$EI\frac{d^2y}{dx^2} + Py = Pa$$

or

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

The solution of above differential equation is:

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + a \qquad (10.3)$$

At $x = 0,\quad y = 0,$

$$\therefore \qquad 0 = C_1 + a$$

or

$$C_1 = -a$$

At $x = 0,\ \frac{dy}{dx} = 0,$

The slope of the section is:

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right)$$

$$\therefore \qquad 0 = 0 + C_2\sqrt{\frac{P}{EI}}$$

It is clear that either

$$C_2 = 0 \qquad \text{or} \qquad l\sqrt{\frac{P}{EI}} = 0$$

But for the crippling load P the value of $l\sqrt{\frac{P}{EI}} \neq 0$.

$$\therefore \qquad C_2 = 0$$

Substituting the values of C_1 and C_2 in Eq. (10.3), we get

$$\therefore \qquad y = -a\cos\left(x\sqrt{\frac{P}{EI}}\right) + a$$

At $\qquad x = l,\ y = a,$

$$\therefore \qquad a = -a\cos\left(l\sqrt{\frac{P}{EI}}\right) + a$$

or

$$\cos\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

or

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

Consider the least practical value:

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$$

or

$$P = \frac{\pi^2 EI}{4l^2}$$

Both Ends are Fixed

Consider the column AB of length l fixed at both the ends. Let P be the crippling load and M be the fixed end moment at A and B.

Bending moment at section $XX = M - Py$

$$\therefore \qquad \frac{EId^2y}{dx^2} = M - Py$$

or

$$EI\,\frac{d^2y}{dx^2} + Py = M$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M}{EI}$$

The solution of above differential equation is:

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M}{P}$$

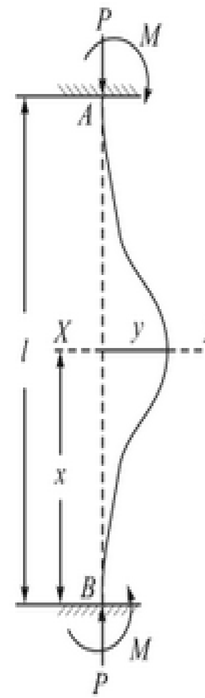
Slope at the section will be

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\ \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2\sqrt{\frac{P}{EI}}\ \cos\left(x\sqrt{\frac{P}{EI}}\right)$$

At B, $x = 0,\ y = 0$

$\therefore$ $0 = C_1 + \frac{M}{P}$

or $C_1 = -\frac{M}{P}$



At B, $x = 0,\ \frac{dy}{dx} = 0 = C_2\sqrt{\frac{P}{EI}}$

$\therefore$ $C_2 = 0$

At A, $x = l,\ y = 0 = -\frac{M}{P}\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{M}{P}$

or $\frac{M}{P}\left(1-\cos\left(l\sqrt{\frac{P}{EI}}\right)\right) = 0$

or $\cos\left(l\sqrt{\frac{P}{EI}}\right) = 1$

or $l\sqrt{\frac{P}{EI}} = 0,\ 2\pi,\ 4\pi,\ 6\pi$

Consider the least practical value:

$$l\sqrt{\frac{P}{EI}} = 2\pi$$

$\therefore$ $$P = \frac{4\pi^2 EI}{l^2}$$

One End is Fixed, Other is Hinged

Consider a column AB of length l fixed at lower end and pinned at upper end. Let P be the crippling load. M is the fixed end moment at support. In order to balance the fixed moment M there will be the horizontal reaction at A.

Bending moment at the section XX is:

$$EI\frac{d^2y}{dx^2} = -Py + H(l - x)$$

or $$EI\frac{d^2y}{dx^2} + Py = H(l - x)$$

or $$EI\frac{d^2y}{dx^2} + Py = H(l - x)$$

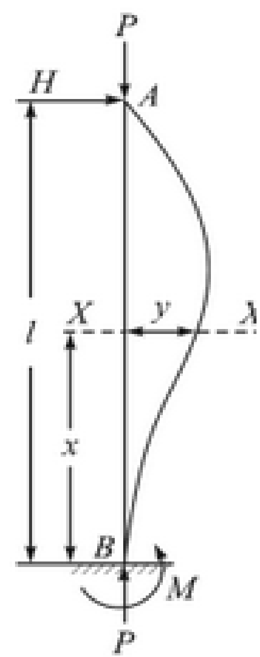
or $$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{H}{EI}(l - x)$$

The solution of the differential equation is:

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x) \quad (10.6)$$

The slope at the section is:

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\ \sin\left(x\sqrt{\frac{P}{EI}}\right) + \cos\left(x\sqrt{\frac{P}{EI}}\right) - \frac{H}{P}$$



At $x = 0$, $y = 0$

$\therefore$ $$0 = C_1 + \frac{H}{P}l$$

or $$C_1 = -\frac{H}{P}l$$

At $x = 0$, $\frac{dy}{dx} = 0 = C_2\sqrt{\frac{P}{EI}} - \frac{H}{P}$

or $$C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}$$

At $x = l$, $y = 0$,

or $$0 = -\frac{H}{P}l\cos\left(l\sqrt{\frac{P}{EI}}\right) + \left(\frac{H}{P}\sqrt{\frac{HI}{P}}\right)\sin\left(l\sqrt{\frac{P}{EI}}\right)$$

Simplifying, we get

$$\tan\left(l\sqrt{\frac{P}{EI}}\right) = l\sqrt{\frac{P}{EI}}$$

The solution to this equation is:

$$l\sqrt{\frac{P}{EI}} = 4.5 \text{ radians}$$

or $$\frac{l^2P}{EI} = (4.5)^2 = 20.25$$

or $$P = \frac{20.25\ EI}{l^2}$$

Approximately, $$20.25 = 2\pi^2$$

or $$P = \frac{2\pi^2 EI}{l^2}$$

Lect-31
Lateral buckling, Critical Load,Slenderness ratio

CRITICAL LOADING

The minimum limiting load at which the column tends to have lateral displacement or tends to buckle is called buckling load or critical load or crippling load.

Lateral buckling: The buckling of column by lateral displacement is known as **Lateral buckling**

SLENDERNESS RATIO

The ratio of the actual length of a column to the least radius of gyration of the column is known as **slenderness ratio**. Mathematically, slenderness ratio is given by

$$\text{Slenderness ratio} = \frac{\text{Actual length}}{\text{Least radius of gyration}}$$

$$= \frac{l}{k} \qquad (10.9)$$

The strength of column depends upon the slenderness ratio and end condition. If the slenderness ratio is increased, the compressive strength of a column decreases as the tendency of buckle is increased.

CLASSIFICATION COLUMN ACCORDING TO SLENDERNESS RATIO

Column can be divided into three types based on their slenderness ratio:

Short column: Column for which slenderness ratio is less than 32 is called **short column**. When short column of uniform cross-sectional area subjected to axial compressive load, the stress induced in the column corresponding to crushing or direct compressive stress. Short column are designed based on crushing stress, as buckling stresses are very small as compared to crushing stress.

$$\frac{l}{k} < 32$$

Medium column: Column for which slenderness ratio is in-between 32 to 120 is called **medium column**. Medium columns are designed based on crushing as well as buckling stress.

$$32 \le \frac{l}{k} \le 120$$

Long column: Column for which slenderness ratio is more than 120 is called **long column**. Long column are designed based on buckling stress, as crushing or direct compressive stresses are very small as compared to buckling stress.

$$\frac{l}{k} > 120$$

CRIPPLING STRESS IN TERMS OF EFFECTIVE LENGTH AND RADIUS OF GYRATION

The moment of inertia I can be expressed in terms of radius of gyration k as:

$$I = Ak^2$$

where A is area of cross-section.

Column will tend to bend in direction of least moment of inertia. So column should be designed using the least value of moment of inertia, then k is the least radius of gyration of the column section. Now, crippling load P in terms of effective length is given by

$$P = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2 E \times Ak^2}{L^2}$$

$$= \frac{\pi^2 E \times A}{\frac{L^2}{k^2}} = \frac{\pi^2 E \times A}{\left(\frac{L}{k}\right)^2}$$

$$\text{Crippling stress} = \frac{\text{Crippling load}}{\text{Area}} = \frac{P}{A}$$

$$= \frac{\pi^2 E \times A}{A\left(\frac{L}{k}\right)^2} \qquad \text{(Substituting the value of } P\text{)}$$

$$= \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2}$$

Lect-32
Torsion in solid and hollow circular shafts, Twisting moment

SHAFT

The shafts are *usually* cylindrical in section, solid or hollow. They are made of mild steel, alloy steel and copper alloys.

Shafts may be subjected to the following loads:

1. Torsional load
2. Bending load
3. Axial load
4. Combination of above three loads.

The shafts are designed on the *basis of strength and rigidity*.

The following values are usually adopted for the design of shaft:

$\sigma = 112$ MN/m^2, the maximum permissible tensile or compressive stress.

$\tau = 56$ MN/m^2, the maximum permissible shear stress.

The ultimate tensile stress for commercial steel shafting may be 315 MN/m^2 for hot rolled and turned low carbon steel and 490 MN/m^2 for cold finished low carbon steel, corresponding stresses at the elastic limit would be about 160 MN/m^2 and 315 MN/m^2 respectively. In shafts with key ways the allowable stresses are 75% of the values given.

TORSION IN A SHAFT

A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft. The torque is equal to the product of the force applied (tangentially to the ends of a shaft) and radius of the shaft. Due to the application of the torques at the two ends, the shaft is subjected to a twisting moment. This causes the shear stresses and shear strains in the material of the shaft.

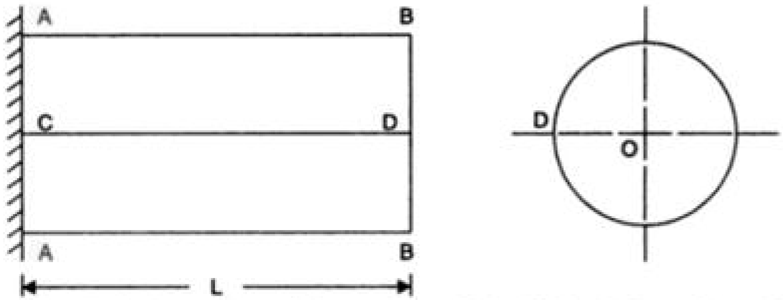
TORSION EQUATION

The *torsion equation* is based on the following **assumptions:**

1. The material of the shaft is uniform throughout.
2. The shaft circular in section remains circular after loading.
3. A plane section of shaft normal to its axis before loading remains plane after the torques have been applied.
4. The twist along the length of shaft is uniform throughout.
5. The distance between any two normal cross-sections remains the same after the application of torque.
6. Maximum shear stress induced in the shaft due to application of torque does not exceed its elastic limit value.

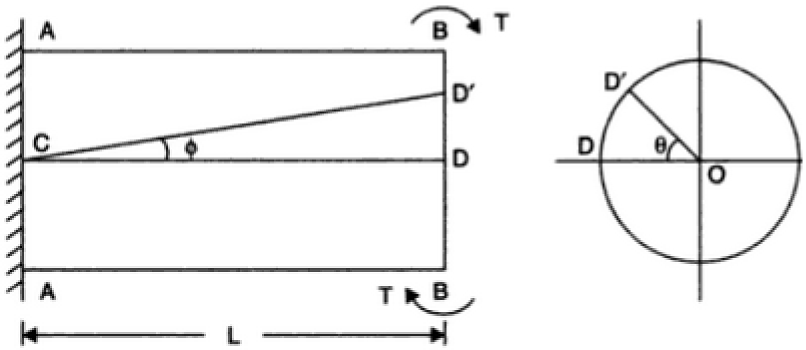
TORSION IN SOLID SHAFT

When a circular shaft is subjected to torsion, shear stresses are set up in the material of the shaft. To determine the magnitude of shear stress at any point on the shaft, consider a shaft fixed at one end *AA* and free at the end *BB* as shown in Fig. 2.96. Let *CD* is any line on the outer surface of the shaft. Now let the shaft is subjected to a torque *T* at the end *BB* as shown in Fig. 2.97. As a result of this torque *T*, the shaft at the end *BB* will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses.



The point *D* will shift to *D′* and hence line *CD* will be deflected to *CD′* as shown in Fig. 2.97 (*a*). The line *OD* will be shifted to *OD′* as shown in Fig. 2.97 (*b*).

Let R = Radius of shaft
L = Length of shaft
T = Torque applied at the end *BB*
f_s = Shear stress induced at the surface of the shaft due to torque T
C = Modulus of rigidity of the material of the shaft
$\phi = \angle DCD'$ also equal to shear strain
$\theta = \angle DOD'$ and is also called angle of twist.



(*a*) (*b*)

Now distortion at the outer surface due to torque T

$= DD'$

∴ Shear strain at outer surface

= Distortion per unit length

$$= \frac{\text{Distortion at the outer surface}}{\text{Length of shaft}} = \frac{DD'}{L}$$

$$= \frac{DD'}{CD} = \tan\phi$$

$$= \phi \qquad \text{(if } \phi \text{ is very small then } \tan\phi \simeq \phi)$$

∴ Shear strain at outer surface,

$$\phi = \frac{DD'}{L} \qquad ...(i)$$

Now from Fig. 2.97 (*b*).

Arc $DD' = OD \times \theta = R\theta$ $\qquad$ ($\because$ $OD = R$ = Radius of shaft)

Substituting the value of DD' in equation (*i*), we get

Shear strain at outer surface

$$\phi = \frac{R \times \theta}{L} \qquad ...(ii)$$

Now the modulus of rigidity (C) of the material of the shaft is given as

$$C = \frac{\text{Shear stress induced}}{\text{Shear strain produced}} = \frac{\text{Shear stress at the outer surface}}{\text{Shear strain at outer surface}}$$

$$= \frac{f_s}{\left(\frac{R\theta}{L}\right)} \qquad \left(\because \text{ From equation } (ii), \text{ shear strain} = \frac{R\theta}{L}\right)$$

$$= \frac{f_s \times L}{R\theta}$$

$$\therefore \quad \frac{C\theta}{L} = \frac{f_s}{R} \qquad ...(2.25)$$

$$\therefore \quad \frac{f_s}{R} = \frac{C\theta}{L} = \frac{q}{r} \; .$$

From equation (*iii*), it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

ASSUMPTION FOR SHEAR STRESS DEVLOPE IN SOLID SHAFT

The derivation of shear stress produced in a circular shaft subjected to torsion, is based on the following assumptions :

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. The shaft is of uniform circular section throughout.
4. Cross-sections of the shaft, which are plane before twist remain plain after twist.
5. All radii which are straight before twist remain straight after twist.

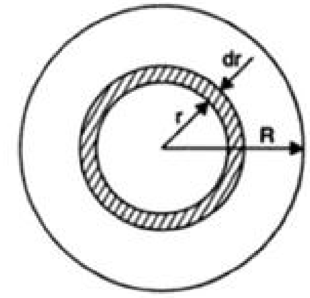
TORQUE TRANSMITTED IN A SOLID SHAFT

The maximum torque transmitted by a circular solid shaft, is obtained from the maximum shear stress induced at the outer surface of the solid shaft. Consider a shaft subjected to a torque T as shown in Fig. 2.98.

Let f_s = Maximum shear stress induced at the outer surface

R = Radius of the shaft

q = Shear stress at a radius 'r' from the centre.



Consider an elementary circular ring of thickness 'dr' at a distance 'r' from the centre as shown in Fig. 2.98. Then the area of the ring,

$$dA = 2\pi r dr$$

From equation (2.26), we have

$$\frac{f_s}{R} = \frac{q}{r}$$

$\therefore$ Shear stress at the radius r,

$$q = \frac{f_s}{R} r = f_s \frac{r}{R}$$

$\therefore$ Turning force on the elementary circular ring

= Shear stress acting on the ring × Area of ring

$= q \times dA$

$= f_s \times \frac{r}{R} \times 2\pi r dr \qquad \left(\because q = f_s \times \frac{r}{R}\right)$

$= \frac{f_s}{R} \times 2\pi r^2 dr$

Now turning moment due to the turning force on the elementary ring,

dT = Turning force on the ring × Distance of the ring from the axis

$$= \frac{f_s}{R} \times 2\pi r^2 dr \times r = \frac{f_s}{R} \times 2\pi r^3 dr \qquad \text{...[2.27 (A)]}$$

$\therefore$ The total turning moment (or total torque) is obtained by integrating the above equation between the limits 0 and R

$$\therefore \quad T = \int_0^R dT = \int_0^R \frac{f_s}{R} \times 2\pi r^3 dr$$

$$= \frac{f_s}{R} \times 2\pi \int_0^R r^3 dr = \frac{f_s}{R} \times 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{f_s}{R} \times 2\pi \times \frac{R^4}{4} = f_s \times \frac{\pi}{2} \times R^3$$

$$= f_s \times \frac{\pi}{2} \times \left(\frac{D}{2}\right)^3 \qquad \left(\because \; R = \frac{D}{2}\right)$$

$$= f_s \times \frac{\pi}{2} \times \frac{D^3}{8} = f_s \times \frac{\pi D^3}{16} = \frac{\pi}{16} f_s D^3 \qquad \text{...(2.28)}$$

TORQUE TRANSMITTED IN A HOLLOW SHAFT

Consider a hollow circular shaft subject to a torque T.
Refer to Fig. 7·3.

Let, R = Outer radius of the shaft,
r = Inner radius of the shaft, and
τ = Shear stress at radius R.

Following the same procedure, we have

dT = Turning moment on the elementary ring

$= \tau_x \,.\, 2\pi x \,.\, dx \,.\, x$

Integrating both sides, we get

$$\int dT = \int_r^R \tau_x \,.\, 2\pi x \,.\, dx \,.\, x$$

but, $\dfrac{\tau_x}{x} = \dfrac{\tau}{R}$ or $\tau_x = \dfrac{\tau}{R} \cdot x$

$$\therefore \quad \int dT = \int_r^R \frac{\tau x}{R} \,.\, 2\pi x \,.\, dx \,.\, x = \frac{2\pi\tau}{R} \int_r^R x^3 \, dx$$

or, $$T = \frac{2\pi\tau}{R} \left| \frac{x^4}{4} \right|_r^R = \frac{\pi}{2} \,.\, \frac{\tau}{R} (R^4 - r^4)$$

or, $$T = \frac{\pi}{16} \tau \left[\frac{D^4 - d^4}{D} \right] \qquad \text{....(\textit{Strength of hollow shaft})}$$

But, $$I_p = \frac{\pi}{32} (D^4 - d^4) = \frac{\pi}{2} (R^4 - r^4)$$

$$\therefore \quad T = \frac{\tau}{R} \cdot I_p$$

or, $$\frac{T}{I_p} = \frac{\tau}{R} = \frac{C\theta}{l} \qquad \text{...(\textit{Torsion equation})}$$

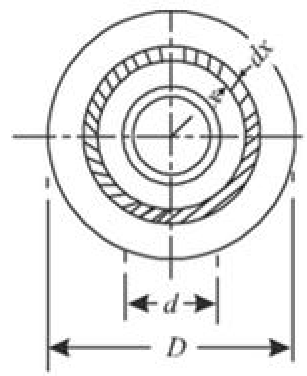


Fig. 7.3

Twisting moment

Twisting moment at section of bar is defined as the algebraic sum of moments at applied couples that lies in one part of the section.

Lect-33
Strain energy in shear and torsion,

Strain energy in shear and torsion in a shaft

Consider a solid circular shaft of length l and radius R, subjected to a torque T producing a twist θ in the length of the shaft (Fig. 8.47).

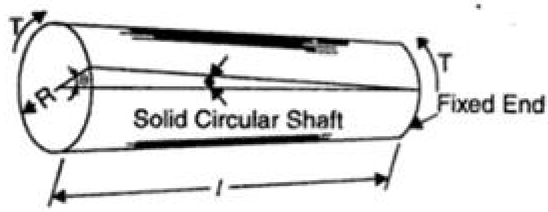


Fig. 8.47

The work done $= \frac{1}{2} T\theta$,

which is stored in the shaft as strain energy.

But $\frac{T}{I_p} = \frac{C\theta}{l} = \frac{\tau}{R}$ ...(Torsion equation)

where T = torque applied,

I_p = polar moment of inertia,

C = modulus of rigidity,

l = length of the shaft, and

τ = maximum shear stress on the surface of the shaft.

$$\therefore \quad T = \frac{\tau \times I_p}{R} \text{ and } \theta = \frac{\tau l}{CR}$$

$$\therefore \text{ Work done } = \frac{1}{2} \frac{\tau \times_p}{R} \times \frac{\tau \times l}{C \times R} = \frac{1}{2} \times \frac{\tau^2}{C} \times \frac{I_p \times l}{R_2}$$

$$\text{Now} \quad I_p = \frac{\pi R^4}{2}$$

$$\therefore \text{ Work done } = \frac{1}{2} \times \frac{\tau^2}{C} \times \frac{\pi R^4 \times l}{2R^2} = \frac{1}{4} \frac{\tau^2}{C} \times \pi R^2 l$$

$$\text{strain energy,} \quad U = \frac{\tau^2}{4C} \times \text{Volume} \quad \text{...(8.20)} \quad (\because \text{Volume} = \pi R^2 l)$$

When the shaft is *hollow*, with an external radius R and internal radius r :

$$\text{Again, work done } = \frac{1}{2} T\theta \text{ and } \theta = \frac{\tau l}{CR} \text{ and } T = \frac{\tau I_p}{R}$$

$$\text{Work done } = \frac{\tau^2}{2C} \times \frac{I_p l}{R_2}$$

$$\text{But} \quad I_p = \frac{\pi}{2}(R^4 - r^4)$$

$$\therefore \text{ Work done } = \frac{\tau^2}{2C} \times \frac{\pi(R^4 - r^4)}{2R^2} = \frac{\tau^2}{4C} \times \frac{\pi l(R^2 + r^2)(R^2 - r^2)}{R^2}$$

$$= \frac{\tau^2}{4C} \times \frac{(R^2 + r^2)}{R^2} \pi (R^2 - r^2) l$$

$$\therefore \quad \text{Strain energy,} \quad U = \frac{\tau^2}{4C} \times \frac{R^2 + r^2}{R^2} \times \text{volume} \quad \text{...(8.21)}$$

Torsional rigidity

From the relation $\frac{T}{I_p} = \frac{C\theta}{l}$

we have $\theta = \frac{Tl}{CI_p}$

Since C, l and I_p are constants for a given shaft, θ *the angle of twist is directly proportional to the twisting moment*. The quantity $\frac{CI_p}{l}$ is known as **torsional rigidity** and is represented by k or μ.

From the above relation, we have

$$k = \frac{CI_p}{l} = \frac{T}{\theta} \qquad \text{...(7·5)}$$

Modulus of rupture

A *modulus of rupture*, corresponding to the modulus of rupture in bending, may be defined as follows:

"*The maximum fictitious shear stress calculated by the torsion formula by using the experimentally found maximum torque* (*i.e.* ultimate torque) *required to rupture a shaft*."

Mathematically, $\tau_r = \frac{T_u R}{I_p}$

where, τ_r = Modulus of rupture in torsion (also called *computed ultimate twisting strength*),

T_u = Ultimate torque at failure, and

R = Outer radius of the shaft.

The above expression for τ_r gives fictitious value of shear stress at the ultimate torque because the torsion formula $\frac{T}{I_p} = \frac{\tau}{R}$ is *not applicable beyond the limit of proportionality*. The actual shear stress at the ultimate torque is quite different from the shearing modulus of rupture because the shear stress does not vary linearly from zero to maximum but it is uniformly distributed at the ultimate torque.

Power transmitted by a shaft

Consider a force F newtons acting tangentially on the shaft of radius R. If the shaft due to this turning moment ($F \times R$) starts rotating at N r.p.m. then,

Work supplied to the shaft/sec.

$= F \times$ distance moved/sec.

$= F \times 2\pi\, RN/60$ Nm/s

or, $$P = \frac{F \times 2\pi\, RN}{60} \text{ watts}$$

$$= \frac{T \times 2\pi\, N}{60 \times 1000} \text{ kW}$$

Hence, $$P = \frac{2\pi NT}{60 \times 1000}$$

Where T is the mean/average torque in Nm.

Lect-34
strength of solid and hollow circular shafts,shaft in series and parallel

strength of solid and hollow circular shafts.

In this case it is assumed that both the shafts have *same length, material, same weight and hence the same maximum shear stress.*

Let, D_S = Diameter of the solid shaft,

d_H = Internal diameter of the hollow shaft,

D_H = External diameter of the hollow shaft,

A_S = Cross-sectional area of solid shaft,

A_H = Cross-sectional area of hollow shaft,

T_S = Torque transmitted by the solid shaft, and

T_H = Torque transmitted by the hollow shaft.

Now,
$$T_S = \tau \cdot \frac{\pi}{16} \times D_S^3$$

$$T_H = \tau \cdot \frac{\pi}{16} \left[\frac{D_H^4 - d_H^4}{D_H} \right]$$

$$\therefore \quad \frac{\text{Strength of hollow shaft}}{\text{Strength of a solid shaft}} = \frac{T_H}{T_S} = \frac{\tau \cdot \frac{\pi}{16} \left[\frac{D_H^4 - d_H^4}{D_H} \right]}{\tau \cdot \frac{\pi}{16} D_S^3}$$

or,
$$\frac{T_H}{T_S} = \frac{D_H^4 - d_H^4}{D_H \cdot D_S^3} \qquad ...(7\cdot7)$$

Let,
$$\frac{D_H}{d_H} = n$$

$\therefore$ $D_H = nd_H$. Substituting it in equation (7·7), we get

$$\frac{T_H}{T_S} = \frac{n^4 d_H^4 - d_H^4}{nd_H D_S^3} = \frac{d_H^4 (n^4 - 1)}{nd_H D_S^3} = \frac{d_H^3 (n^4 - 1)}{n D_S^3} \qquad ...(7\cdot8)$$

As the weight, material and length of both the shafts are same,

$\therefore$ Cross-sectional area of solid shaft = Cross-sectional area of hollow shaft $A_S = A_H$

$$\therefore \quad \frac{\pi}{4} D_S^2 = \frac{\pi}{4} (D_H^2 - d_H^2) \text{ or } D_s = \sqrt{D_H^2 - d_H^2}$$

or,
$$D_S^3 = (D_H^2 - d_H^2) \sqrt{D_H^2 - d_H^2}$$

$$D_S^3 = (n^2 d_H^2 - d_H^2) \sqrt{n^2 d_H^2 - d_H^2}$$

$$D_S^3 = d_H^3 (n^2 - 1) \sqrt{n^2 - 1} \qquad ...(7\cdot9)$$

Substituting the value of D_S^3 in equation (7·8), we get

$$\frac{T_H}{T_S} = \frac{d_H^3\ (n^4 - 1)}{n\ d_H^3\ (n^2 - 1)\sqrt{n^2 - 1}}$$

$$= \frac{(n^2 + 1)\ (n^2 - 1)}{n\ (n^2 - 1)\sqrt{n^2 - 1}} = \frac{n^2 + 1}{n\sqrt{n^2 - 1}} \qquad \text{...(7·10)}$$

Since $D_H > d_H$ and $= \frac{D_H}{d_H} > n$, it is obvious that the value of 'n' is greater than unity.

∴ Suppose, $n = 2$

Then, $\frac{T_H}{T_S} = \frac{2^2 + 1}{2\sqrt{2^2 - 1}} = 1{\cdot}44$

This shows that the torque transmitted by the hollow shaft is greater than the solid shaft, thereby proving that the hollow shaft is stronger than the solid shaft.

(*b*) Comparison by weight:

In this case it is assumed that both the shafts have the *same length and material*. Now, if the torque applied to both shafts is same, then, the maximum shear stress will also be same in both the cases.

Now, $\frac{\text{Weight of hollow shaft}}{\text{Weight of solid shaft}} = \frac{W_H}{W_S} = \frac{A_H}{A_S}$

$$= \frac{\frac{\pi}{4}(D_H^2 - d_H^2)}{\frac{\pi}{4}D_S^2} = \frac{D_H^2 - d_H^2}{D_S^2} \qquad \text{...(7·11)}$$

Let, $\frac{D_H}{d_H} = n$

∴ $D_H = nd_H$ and substituting this value in equation (7·10), we get

$$\frac{W_H}{W_S} = \frac{n^2\ d_H^2 - d_H^2}{D_S^2} = \frac{d_H^2\ (n^2 - 1)}{D_S^2} \qquad \text{...(7·12)}$$

Torque applied in both the cases is same *i.e.*, $T_S = T_H$

$$\tau \,.\, \frac{\pi}{16}\, D_s^3 = \tau \,.\, \frac{\pi}{16}\left[\frac{D_H^4 - d_H^4}{D_H}\right]$$

$$D_s^3 = \frac{D_H^4 - d_H^4}{D_H} = \frac{n^4\ d_H^4 - d_H^4}{n\,d_H} = \frac{d_H^3\ (n^4 - 1)}{n}$$

∴ $$D_s = d_H\left[\frac{n^4 - 1}{n}\right]^{1/3}$$

or, $$D_s^2 = d_H^2\left[\frac{n^4 - 1}{n}\right]^{2/3} \qquad \text{...(7·13)}$$

Substituting the value of D_S^2 in equation (7·12), we have

$$\frac{W_H}{W_S} = \frac{d_H^2\ (n^2 - 1)}{d_H^2 \left(\frac{n^4 - 1}{n}\right)^{2/3}} = \frac{(n^2 - 1)\ n^{2/3}}{(n^4 - 1)^{2/3}} \quad ...(7{\cdot}14)$$

If, $n = 2$ then, $$\frac{W_H}{W_S} = \frac{(2^2 - 1) \times (2)^{2/3}}{(2^4 - 1)^{2/3}} = 0{\cdot}7829$$

which shows that for *same material, length and given torque, weight of hollow shaft will be less. So, hollow shafts are economical compared to solid shafts as regards torsion.*

Shaft are in series and parallel

In order to form a composite shaft sometimes two shafts are connected in series. In such cases, *each shaft transmits the same torque. The angle of twist is the sum of the angle of twist of the two shafts connected in series.*

Thus, total angle of twist is given by

$$\theta = \theta_1 + \theta_2 = \frac{Tl_1}{C_1 I_{P_1}} + \frac{Tl_2}{C_2 I_{P_2}} = T\left[\frac{l_1}{C_1 I_{P_1}} + \frac{l_2}{C_2 I_{P_2}}\right] \quad ...(7{\cdot}15)$$

where, T = Torque transmitted by each shaft,

l_1, l_2 = Respective lengths of the two shafts,

C_1, C_2 = Respective moduli of rigidity, and

I_{P_1}, I_{P_2} = Respective polar moment of intertias.

When shafts are made of same material,

$$C_1 = C_2 = C \text{ say}$$

Then, $$\theta = \frac{T}{C}\left[\frac{l_1}{I_{P_1}} + \frac{l_2}{I_{P_2}}\right]$$

Here, the driving torque is applied at one end and the resisting torque at the other.

Parallel

The shafts are said to be in *parallel when the driving torque is applied at the junction of the shafts and the resisting torque is at the other ends of the shafts. Here, the angle of twist is same for each shaft, but the applied torque is divided between the two shafts.*

i.e. $$\theta_1 = \theta_2$$

or, $$\frac{T_1 l_1}{C_1 I_{P_1}} = \frac{T_2 l_2}{C_2 I_{P_2}} \quad ...(7{\cdot}16)$$

and, $$T = T_1 + T_2 \quad ...(7{\cdot}17)$$

If the shafts are made of same material

$$C_1 = C_2$$

Then, $$\frac{T_1 l_1}{I_{P_1}} = \frac{T_2 l_2}{I_{P_2}} \text{ or } \frac{T_1}{T_2} = \frac{I_{P_1} l_2}{I_{P_2} l_1} \quad ...[7{\cdot}18\ (a)]$$

When torque is shared equally by both the shafts

$$T_1 = T_2, \text{ then, } I_{P_1} l_2 = I_{P_2} l_1 \quad ...[7{\cdot}18\ (b)]$$

Lect-35
Stresses due to combined bending and torsion

Stresses due to combined bending and torsion

Problems in this category frequently occur in circular-section shafts, particularly those of ships. In such cases, the combination of shearing stresses due to torsion and direct stresses due to bending causes a complex system of stress.

Consider a circular-section shaft subjected to a combined bending moment M and a torque T, as shown in Figure 7.13.

The largest bending stresses due to M will be at both the top and the bottom of the shaft, and the combined effects of bending stress and shear stress at those positions will be as shown in Figure 7.14.

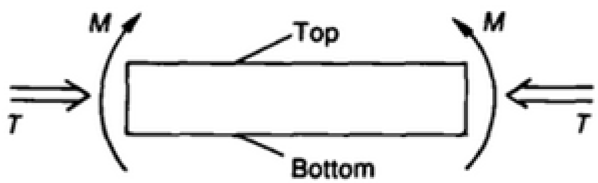


Figure 7.13 Shaft under combined bending and torsion

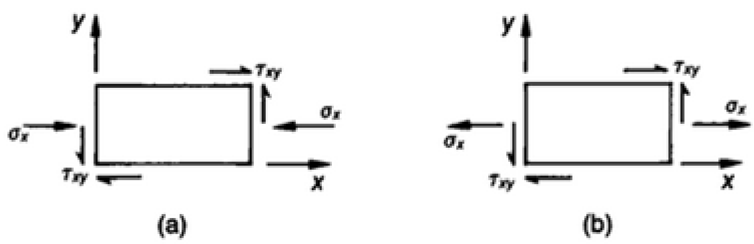


Figure 7.14 Complex stress system due to M and T: (a) top of shaft (looking down); (b) bottom of shaft (looking up)

Now σ_x is entirely due to M, and τ_{xy} is entirely due to T, so that

$$\sigma_x = M\left(\frac{64}{\pi d^4}\right)\frac{d}{2}$$

$$\sigma_x = \frac{32M}{\pi d^3} \qquad (7.15)$$

and

$$\tau_{xy} = T\left(\frac{32}{\pi d^4}\right)\frac{d}{2}$$

$$\tau_{xy} = \frac{16T}{\pi d^3} \tag{7.16}$$

From equilibrium considerations,

$$\sigma_y = 0$$

Substituting equations (7.15) and (7.16) into equations (7.6) and (7.7),

$$\sigma_1, \sigma_2 = \frac{16M}{\pi d^3} \pm \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{16}{\pi d^3}\left(M \pm \sqrt{M^2 + T^2}\right) \tag{7.17}$$

Similarly,

$$\hat{\tau} = \frac{16}{\pi d^3}\left(M^2 + T^2\right) \tag{7.18}$$

Equations (7.17) and (7.18) can also be written in the form

$$\sigma_1, \sigma_2 = \frac{32M_e}{\pi d^3} \tag{7.19}$$

and

$$\hat{\tau} = \frac{16T_e}{\pi d^3} \tag{7.20}$$

where

M_e = equivalent bending moment

$$= \tfrac{1}{2}\left(M \pm \sqrt{M^2 + T^2}\right) \tag{7.21}$$

and

T_e = equivalent torque

$$= \sqrt{M^2 + T^2} \tag{7.22}$$

Lect-36
Stresses due to combined bending and Twisting

Strength of shafts in combined bending and twisting.

Consider $M_{e,z}$ to be the equivalent bending which acting alone will develop the maximum tensile stress equal to σ_1, then

$$\sigma_1 = \frac{M_{e,z}}{(\pi D^3/32)} = \frac{32M_{e,z}}{\pi D^3} \quad (20.46)$$

$$\frac{32M_{e,z}}{\pi D^3} = \left(\frac{16}{\pi D^3}\right)\left[M_z \pm \sqrt{M_z^2 + T_z^2}\right]$$

or

$$M_{e,z} = \left(\frac{1}{2}\right)\left[M_z \pm \sqrt{M_z^2 + T_z^2}\right] \quad (20.47)$$

Similarly, if $T_{e,z}$ is the equivalent twisting moment which acting alone develops the maximum shearing stress equal to τ_{max}, then

$$\tau_{max} = \frac{T_{e,z}}{(\pi D^3/16)} = \frac{16T_{e,z}}{\pi D^3} \quad (20.48)$$

Therefore,

$$\frac{16T_{e,z}}{\pi D^3} = \left(\frac{16}{\pi D^3}\right)\left[\sqrt{M_z^2 + T_z^2}\right]$$

or

$$T_{e,z} = \left[\sqrt{M_z^2 + T_z^2}\right] \quad (20.49)$$

Therefore, the ratio of the maximum shearing stress to the principal stress is:

$$\frac{\tau_{max}}{\sigma_1} = \frac{T_{e,z}}{(\pi D^3/16)} \times \frac{(\pi D^3/32)}{M_{e,z}}$$

$$= \frac{T_{e,z}}{2M_{e,z}} = \frac{\sqrt{M_z^2 + T_z^2}}{M_z + \sqrt{M_z^2 + T_x^2}} \quad (20.50)$$

Lect-37

Types springs, spring material

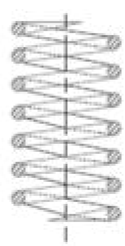
Introduction

A spring is defined as an elastic body, whose function is to distort when loaded and to recover its original shape when the load is removed. The various important applications of springs are as follows:

1. To cushion, absorb or control energy due to either shock or vibration as in car springs, railway buffers, air-craft landing gears, shock absorbers and vibration dampers.
2. To apply forces, as in brakes, clutches and spring loaded valves.
3. To control motion by maintaining contact between two elements as in cams and followers.
4. To measure forces, as in spring balances and engine indicators.
5. To store energy, as in watches, toys, etc.

Types of springs:

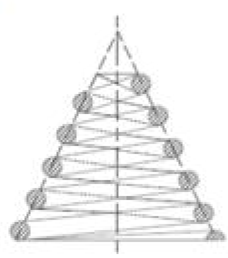
1. Helical springs. The helical springs are made up of a wire coiled in the form of a helix and is primarily intended for compressive or tensile loads.



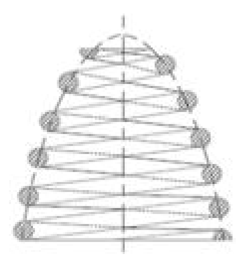
(*a*) Compression helical spring.

(*b*) Tension helical spring.

2. Conical and volute springs. The conical and volute springs, as shown in Fig. 23.2, are used in special applications where a telescoping spring or a spring with a spring rate that increases with the load is desired

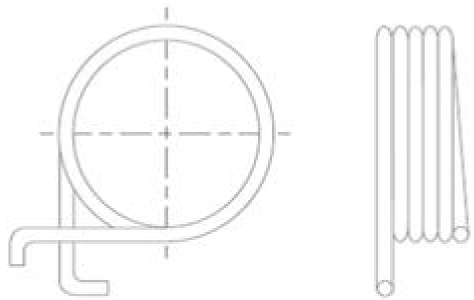


(*a*) Conical spring.

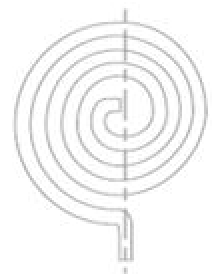


(*b*) Volute spring.

3. Torsion springs. These springs may be of **helical** or **spiral** type as shown in Fig. The **helical type** may be used only in applications where the load tends to wind up the spring and are used in various electrical mechanisms.

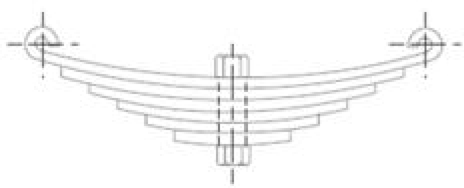


(*a*) Helical torsion spring.

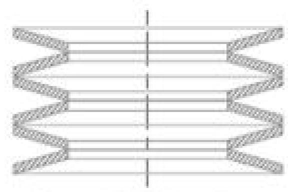


(*b*) Spiral torsion spring.

4. Laminated or leaf springs. The laminated or leaf spring (also known as **flat spring** or **carriage spring**) consists of a number of flat plates (known as leaves) of varying lengths held together by means of clamps and bolts.



Laminated or leaf springs.



Disc or bellevile springs.

5. Disc or bellevile springs. These springs consist of a number of conical discs held together against slipping by a central bolt or tube.

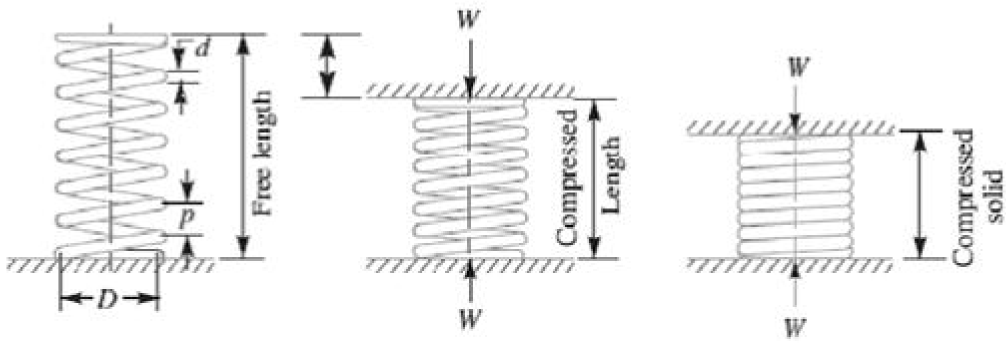
6. Special purpose springs. These springs are air or liquid springs, rubber springs, ring springs etc. The fluids (air or liquid) can behave as a compression spring. These springs are used for special types of application only.

Terms used in Compression Springs

1. Solid length. When the compression spring is compressed until the coils come in contact with each other, then the spring is said to be **solid.**

Solid length of the spring, $L_s = n'.d$ where n' = Total number of coils, and d = Diameter of the wire.

2. Free length. The free length of a compression spring, as shown in Fig., is the length of the spring in the free or unloaded condition.



Free length of the spring,

L_F = Solid length + Maximum compression + *Clearance between adjacent coils (or clash allowance)

$= n'.d + \delta_{max} + 0.15\ \delta_{max}$

3. Spring index. The spring index is defined as the ratio of the mean diameter of the coil to the diameter of the wire. Spring index, C = D / d where D = Mean diameter of the coil, and d = Diameter of the wire.

4. Spring rate. The spring rate (or stiffness or spring constant) is defined as the load required per unit deflection of the spring. Mathematically, Spring rate, k = W / δ where W = Load, and δ = Deflection of the spring.

5. Pitch. The pitch of the coil is defined as the axial distance between adjacent coils in uncompressed state. Mathematically, Pitch of the coil,

$$p = \frac{\text{Free Length}}{n' - 1}$$

Lect-38
Stress developed in springs,Wahl's factor

Stresses in Helical Springs of Circular Wire

Consider a helical compression spring made of circular wire and subjected to an axial as shown in Fig.(*a*).

Let D = Mean diameter of the spring coil,

d = Diameter of the spring wire,

n = Number of active coils,

G = Modulus of rigidity for the spring material,

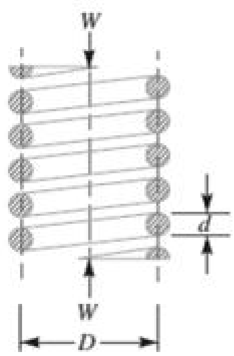
W = Axial load on the spring,

τ = Maximum shear stress induced in the wire,

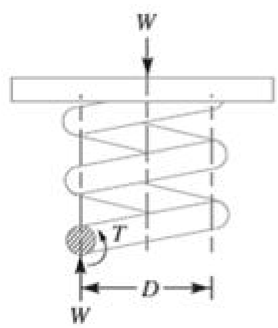
C = Spring index = D/d,

p = Pitch of the coils, and

δ = Deflection of the spring, as a result of an axial load W.



a) Axially loaded helical spring.



(*b*) Free body diagram showing that wire is subjected to torsional shear and a direct shear.

Now consider a part of the compression spring as shown in Fig. (*b*). The load W tends to rotate the wire due to the twisting moment (T) set up in the wire. Thus torsional shear stress is induced in the wire.

A little consideration will show that part of the spring, as shown in Fig.(*b*), is in equilibrium under the action of two forces W and the twisting moment T. We know that the twisting moment,

$$T = W \times \frac{D}{2} = \frac{\pi}{16} \times \tau_1 \times d^3$$

$$\tau_1 = \frac{8W.D}{\pi d^3} \qquad ...(i)$$

The torsional shear stress diagram is shown in Fig. (*a*).

In addition to the torsional shear stress (τ_1) induced in the wire, the following stresses also act on the wire:

1. Direct shear stress due to the load *W*, and

2. Stress due to curvature of wire.

We know that the resultant shear stress induced in the wire,

$$\tau = \tau_1 \pm \tau_2 = \frac{8W.D}{\pi\, d^3} \pm \frac{4W}{\pi\, d^2}$$

Maximum shear stress induced in the wire,

= Torsional shear stress + Direct shear stress

$$= \frac{8W.D}{\pi\, d^3} + \frac{4W}{\pi\, d^2} = \frac{8W.D}{\pi\, d^3}\left(1 + \frac{d}{2D}\right)$$

$$= \frac{8\,W.D}{\pi\, d^3}\left(1 + \frac{1}{2C}\right) = K_S \times \frac{8\,W.D}{\pi\, d^3} \quad \text{...(iii)}$$

... (Substituting $D/d = C$)

where K_S = Shear stress factor $= 1 + \dfrac{1}{2C}$

∴ Maximum shear stress induced in the wire,

$$\tau = K \times \frac{8\,W.D}{\pi\, d^3} = K \times \frac{8\,W.C}{\pi\, d^2} \quad \text{...(iv)}$$

where $K = \dfrac{4C - 1}{4C - 4} + \dfrac{0.615}{C}$

Lect-39
Deflection in helical spring , Springs in series and parallel

Deflection of Helical Springs of Circular Wire

Total active length of the wire,

$$l = \text{Length of one coil} \times \text{No. of active coils} = \pi D \times n$$

Let θ = Angular deflection of the wire when acted upon by the torque T.

$\therefore$ Axial deflection of the spring,

$$\delta = \theta \times D/2 \quad \text{...(i}$$

We also know that

$$\frac{T}{J} = \frac{\tau}{D/2} = \frac{G.\theta}{l}$$

$$\therefore \quad \theta = \frac{T.l}{J.G} \quad \text{...}\left(\text{considering } \frac{T}{J} = \frac{G.\theta}{l}\right)$$

where J = Polar moment of inertia of the spring wire

$$= \frac{\pi}{32} \times d^4, \; d \text{ being the diameter of spring wire.}$$

and G = Modulus of rigidity for the material of the spring wire.

Now substituting the values of l and J in the above equation, we have

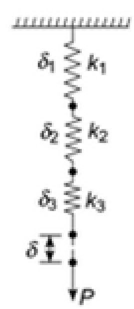
$$\theta = \frac{T.l}{J.G} = \frac{\left(W \times \frac{D}{2}\right)\pi D.n}{\frac{\pi}{32} \times d^4 G} = \frac{16 W.D^2.n}{G.d^4} \quad \text{...(ii}$$

Springs in series and parallel

Several close coiled helical springs may be used in two distinct combinations to carry a single load or combination of loads that can be converted to loads carried along the axes of individual springs. The two combinations are described here.

Series Spring System The springs in this system join end to end and the same load is carried by each spring. The joining of springs in series system is shown in Fig. 12.6. Apparently the deflection δ at the load point is the sum of individual deflections. i.e.

$$\delta = \delta_1 + \delta_2 + \delta_3 \quad \text{(i)}$$



Spring In series

For an axial load P and if stiffnesses or *spring constants* of the springs are k_1, k_2, k_3, (i) can be written as

$$\delta = \frac{P}{k_1} + \frac{P}{k_2} + \frac{P}{k_3} \tag{ii}$$

or

$$\frac{\delta}{P} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

i.e.

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \tag{12.16}$$

Here k_e is the stiffness of a single spring equivalent to the springs connected in series.

Parallel Spring System Parallel connected springs are shown in Fig. 12.7. In this case the deflection of each spring is same while the load carried by each is different. The equivalent spring is that which carries the sum of loads of individual springs and also deflects through the same distance as others in the system. Thus if loads carried by individual springs are P_1, P_2, P_3, the total load carried by system is P, then

$$P = P_1 + P_2 + P_3 \tag{iii}$$

If deflection of the system and its members is δ, then

$$\frac{P}{\delta} = \frac{P_1}{\delta} + \frac{P_2}{\delta} + \frac{P_3}{\delta}$$

or

$$k_e = k_1 + k_2 + k_3 \tag{12.17}$$

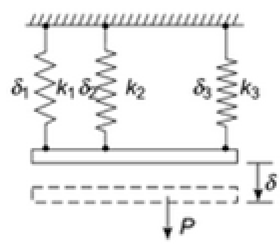


Fig. 12.7 Springs in parallel

where k_e, k_1, k_2 and k_3 have the same meaning as stated in case of series system. The Eqn. (12.17) states that the stiffness of a parallel connected system of helical springs is the sum of stiffnesses of individual springs.

Reference Books:

1. Strength Of Material by Sadhu Singh.
2. Strength Of Material by S S Ratan.
3. Strength Of Material By R K Rajput.
4. Strength Of Material by R K Bansal.
5. Lecture notes from IITs and IISc.